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# IDEALITY AND SUBIDEALITY FROM A COMPUTATIONAL POINT OF VIEW

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## 1 Introduction

Why should Law need automated proof systems? The answer to this question implies an answer to the following question: Is logic needed in Law? In fact it has been argued that logics are useless for Law (see, for example, Kelsen 1989). We believe that logic, and deontic logics in particular — but also modal logics — have a role to play in Law; for example if one wants to study what the relationships are among the various degrees of adjudication in Italian Law, one should note that they give rise to a transitive, irreflexive and finite structure, which is the frame of the modal logic of *provability* $GL$; one of the most important properties of such a logic is that no system, (no court) in this frame, could claim its own correctness without becoming incorrect (Boolos 1993, Smullyan 1988), but the correctness of a lower court can be established by a higher one. This example shows that the study of modal logic can help in finding certain already known properties of legal systems. Moreover, each time we are dealing with the notions of Obligation and Permission, and we are interested in the study of their mutual relationship, we can arrange them into a deontic framework, thus producing a certain kind of deontic logic. Finally a hint for the use of logic in legal reasoning is given, for example in the Italian case, by the law itself; in fact article 192, 1° comma of the "Italian code of criminal procedure" prescribes that the judges state the reasons of their adjudication; moreover several other articles of the same code, state: when evidence is valid, how evidence should be used in order to lead to an adjudication, etc. On this basis the "Italian code of criminal procedure" can be thought of as a deductive system where its articles act as the inference rules, whereas the articles of the "Italian code of criminal law" are the axioms.

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What does a proof system do? A proof system can work in two ways. The first of them consists of producing admissible steps one after the other according to the inference rules; in this way each step is guaranteed to be correct, but we are not led to the goal we want to prove. The other one consists of verifying whether a conclusion follows from given premises, i.e., if the adjudication follows logically from the evidence, mainly by refuting the negation of the conclusion.

The system we propose is based on the logic of ideality and subideality developed by Jones and Pörn, and it verifies in the above mentioned logical framework whether a given conclusion follows from given premises. Moreover, due to its basic control structure it can also be used as an analytic direct proof system.

## 2 Ideal and Subideal Deontic Logic

This logic has been developed in (Jones and Pörn 1985;1986) in order to have a system which permits both factual and deontic detachment; moreover it is possible to define several types of obligation, i.e., ideal obligation and subideal obligation, thus avoiding the drawback that "actual" obligation collapses in the logical necessity of the framework we are using (deontic necessity).

As pointed out by Kelsen

> Jurisprudence, by describing the validity of a law system, does not assert what happens regularly, but what ought to happen according to a given law system (Kelsen 1989, 458).

Jones and Pörn's (1985, 1986) deontic logic $DL$ has been devised for dealing with ideal as well as sub-ideal situations, i.e., situations which admit some degree of violation of what is ideally the case. Formally it is an extension of standard deontic logic ($SDL$), which is a normal $KD$ system according to Chellas' (1981) classification, and which incorporates, besides the normal deontic operators $\mathsf{O}^i$ and $\mathsf{P}^i$, the deontic operators $\mathsf{O}^s$ and $\mathsf{P}^s$. $\mathsf{O}^i$ and $\mathsf{P}^i$ retain their usual reading. $\mathsf{O}^i A$ ($\mathsf{P}^i A$), at a world $w$, mean: $A$ holds in all (some) of $w$'s deontically ideal versions. $\mathsf{O}^s A$ ($\mathsf{P}^s A$), at a world $w$, mean: $A$ holds in all (some) of $w$'s sub-ideal versions. $DL$ allows us to define the following notions:

- $\mathsf{N}_D A =_{df} (\mathsf{O}^i A \wedge \mathsf{O}^s A)$ (*Deontic Necessity*)
- $\mathsf{O}_\top A =_{df} (\mathsf{O}^i A \wedge \mathsf{P}^s \neg A)$ (*Ought*)

Since $DL$ is a straightforward extension of $SDL$ both $\mathsf{O}^i$, $\mathsf{P}^i$ and $\mathsf{O}^s$, $\mathsf{P}^s$ behave as normal $KD$-modalities. Models for $DL$ are thus structures:

$$M = \langle W, R_i, R_s, v \rangle$$

where $R_i$, $R_s \subseteq W \times W$ are serial (not reflexive) relations on $W$ (intuitive reading: $wR_iv$ = $v$ is an ideal version of $w$, $wR_sv$ = $v$ is a sub-ideal version of $w$), subject to the following conditions:

**C1:** $R_i \cap R_s = \emptyset$

**C2:** $\{\langle w,w\rangle : w \in W\} \subseteq R_i \cup R_s$

This means that there cannot exist ideal worlds that are also sub-ideal, and every world is either ideal or sub-ideal relative to itself (notice that this introduces some form of reflexivity in the model); $v$ is as usual with the following clauses for $\mathsf{O}^i$ and $\mathsf{O}^s$ respectively

$$\models_w \mathsf{O}^i A \Leftrightarrow \forall v \in W : wR_iv, \models_v A$$

$$\models_w \mathsf{O}^s A \Leftrightarrow \forall v \in W : wR_sv, \models_v A$$

*Remark* 1. Condition **C2** has been dropped in Jones 1991, so that possible worlds can be both ideal and subideal with respect to themselves; however, when condition **C2** may be parametrized with respect to content matters, see Epstein 1990; but this may lead to a more satisfactory solution; in fact a section of the "Italian code of criminal procedure" concerns the connected crimes: in a criminal trial pieces of evidence from other trials are examined if and only if they are judged to be relevant (connected) to the subject of the trial; a parking fine, will not usually be considered relevant in an adjudication for murder.

## 3 The System $KEM$

In this section we shall present $KEM$ in its barest outline. We first recall some basic notions. We shall use the letters $X, Y, Z, \ldots$ to denote arbitrary signed formulas ($S$-formulas), i.e., formulas of the forms $SA$ where $S \in \{T, F\}$. As usual $X^C$ will be used to denote the *conjugate* of $X$, i.e., the result of changing $S$ to its opposite (with the exception of the following $S$-formulas[1]: $T\Box A, F\Diamond A, F\Box A$ and $F\Diamond A$, which also have $T\Diamond\neg A, F\Box\neg A, F\Diamond\neg A, T\Box\neg A$ respectively as their conjugates). Two $S$-formulas $X, Z$ such that $Z = X^C$, will be called *complementary*. As we have already said, $KEM$ approach requires us to work with "world" labels. A "world" label is either a constant or a variable "world" symbol or a "structured" sequence of world-symbols we call a "world-path". Intuitively, constant and variable world-symbols stand for worlds and sets of worlds respectively, while a world-path conveys information about access between the worlds in it. We attach labels to $S$-formulas to yield *labelled signed formulas* (*LS-formulas*), i.e., pairs of the form $X, i$ where $X$ is an $S$-formula and $i$ is a label. An $LS$-formula $SA, i$ means, intuitively, that $A$ is true (false) at the (last) world (on the path represented by) $i$. In the course of proof search, labels are manipulated in a way closely related to the semantics of modal operators and "matched" using a (specialized, logic-dependent) unification algorithm. That two world-paths $i$ and $k$ are unifiable means, intuitively, that they virtually represent the same path, i.e., any world which you could arrive at by path $i$ could be reached by path $k$ and vice versa. $LS$-formulas whose labels are unifiable turn out to be true (false) at the same world(s) relative to the accessibility relation that holds in the appropriate class of models. In particular two $LS$-formulas $X, X^C$

whose labels are unifiable stand for formulas which are contradictory "in the same world". These ideas are formalized as follows.

### 3.1 *Label Formalism*

To treat $DL$ we need three kinds of label world symbols

- Universal $\Phi_W = \{W_1, W_2, \cdots\}$ and $\Phi_w = \{w_1, w_2, \cdots\}$
- Ideal $\Phi_D = \{D_1, D_2, \cdots\}$ and $\Phi_d = \{d_1, d_2, \cdots\}$
- Subideal $\Phi_S = \{S_1, S_2, \cdots\}$ and $\Phi_s = \{s_1, s_2, \cdots\}$

Here the universal world labels denote worlds for which we do not have enough information to specify whether they are ideal or subideal. Let us now define the set of variable world symbols and constant world symbols respectively:

$$\begin{aligned} \Phi_V &= \Phi_W \cup \Phi_D \cup \Phi_S \text{ and} \\ \Phi_C &= \Phi_w \cup \Phi_d \cup \Phi_s. \end{aligned}$$

On this basis the set $\Im$ is now defined as

$$\begin{aligned} \Im &= \bigcup_{1 \leq i} \Im_i \text{ where } \Im_i \text{ is :} \\ & \Im_1 = \Phi_C \cup \Phi_V; \\ & \Im_2 = \Im_1 \times \Phi_C; \\ & \Im_{n+1} = \Im_1 \times \Im_n. \end{aligned}$$

In other words a world-label is either (i) an element of the set $\Phi_C$, or (ii) an element of the set $\Phi_V$, or (iii) a path term $(k', k)$ where (iiia) $k' \in \Phi_C \cup \Phi_V$ and (iiib) $k \in \Phi_C$ or $k = (m', m)$ where $(m', m)$ is a label. It is worth noting that such a representation of labels captures the precise meaning of the accessibility relation; in fact, the label $(w_2, (W_1, w_1))$ denotes a path which leads to a world $(w_2)$ accessible from all the worlds accessible from the world denoted by $w_1$. If labels were sequences of constants and variables, their reading would be ambiguous: i.e., what does a label such as $\langle W_1, w_2, w_1 \rangle$ stand for? Does it mean that $w_1$ sees all the worlds accessible from $w_2$, or does it have the same meaning as $(W_1, (w_2, w_1))$? The two possible readings of a label written as a sequence, give rise to different accessibility relations.

A bit of terminology. For any label $i = (k', k)$ we call $k'$ the *head* of $i$, $k$ the *body* of $i$, and denote them by $h(i)$ and $b(i)$ respectively. Notice that these notions are recursive: if $b(i)$ denotes the body of $i$, then $b(b(i))$ will denote the body of $b(i)$, $b(b(b(i)))$ will denote the body of $b(b(i))$; and so on. For example, if $i$ is $(w_4, (W_3, (w_3, (W_2, w_1))))$, then $b(i) = (W_3, (w_3, (W_2, w_1)))$, $b(b(i)) = (w_3, (W_2, w_1))$, $b(b(b(i))) = (W_2, w_1)$, $b(b(b(b(i)))) = w_1$. We call each of $b(i)$,$b(b(i))$, etc., a *segment* of $i$. Let $s(i)$ denote any segment of $i$ (obviously, by definition every segment $s(i)$ of a label $i$ is a label); then $h(s(i))$ will denote the head of $s(i)$.

For any label $i$, we define the length of $i$, $l(i)$, as the number of world-symbols in $i$, i.e., $l(i) = n \Leftrightarrow i \in \Im_n$.

We shall use $s^n(i)$ to denote the segment of $i$ whose length is $n$

### 3.2 *Unification Schemes*

$KEM$'s label unification scheme involves two kinds of unifications, respectively "high" and "low". "High" unifications are meant to mirror specific accessibility constraints and they are used to build "low" unifications, which account for the full range of conditions governing the appropriate accessibility relation. We then begin by defining the basic notion of "high" unification. First we define a substitution in the usual way as a function

$$\begin{aligned} \sigma \quad &: \quad \Phi_V^0 \longrightarrow \Im^- \\ &: \quad \Phi_V^i \longrightarrow \Im^i, (1 \leq i \leq n). \end{aligned}$$

where $\Im^- = \Im - \Phi_V$. For two labels $i$, $k$ and a substitution $\sigma$, if $\sigma$ is a unifier of $i$ and $k$ then we shall say that $i$ and $k$ are $\sigma$-unifiable. We shall (somewhat unconventionally) use $(i,k)\sigma$ to denote both that $i$ and $k$ are $\sigma$-unifiable and the result of their unification.

$$(i,k)\sigma = \begin{cases} \sigma i = \sigma k & l(i) = l(k) = 1 \\ ((h(i), h(k))\sigma, (b(i), b(k))\sigma) & \end{cases}$$

In order to get the appropriate unifications we need to define the following substitution acting as $\sigma$ for universal world symbols:

$$\sigma^{\#}\Phi_W = \sigma\Phi_W$$

and as follows for ideal an sub-ideal world symbols:

$$\begin{aligned} \sigma^{\#} \quad &: \quad \Phi_S \rightarrow \Phi^s \\ &: \quad \Phi_D \rightarrow \Phi^d \end{aligned}$$

where

$$\Phi^d = \{i^r \in \Phi_C : r = d\} \qquad \Phi^s = \{i^r \in \Phi_C : r = s\}$$

$\Phi^d, \Phi^s$ denote the set of worlds that are respectively an ideal and a subideal version of themselves.

According to the above substitutions we define the

$$(i,k)\sigma^R = \begin{cases} (s^{l(k)}(i),k)\sigma & l(i) > l(k), h(k) \in \Phi_C \text{ e} \\ & \forall m > l(k), (i^m, h(k))\sigma^\# = (i^{l(k)}, h(k))\sigma \\ (i, s^{l(i)}(k))\sigma & l(k) > l(i), h(i) \in \Phi_C \text{ e} \\ & \forall m > l(i), (h(i), k^m)\sigma^\# = (h(i), k^{l(i)})\sigma \end{cases} \tag{1}$$

For example the labels

$$(D_1,(w_2^i,w_1)) \qquad\qquad (w_2^i,w_1)$$

$\sigma^R$-unify since

$$w_2^i = (D_1, w_2^i)\sigma^\# = (w_2^i, w_2^i)\sigma^+$$

and obviously $(w_1, w_1)\sigma^+$.

We are now able to characterize $DL$ by the notion of $\sigma_{DL}$-unification:

$$(i,k)\sigma_{DL} = \begin{cases} (i,k)\sigma^+ \\ (i,k)\sigma^R \end{cases} \tag{2}$$

from which follows

$$(i,k)\sigma_{DL} = \begin{cases} (c^n(i), c^m(k))\sigma^{DL} \\ (i,k)\sigma^{DL} \end{cases} \tag{$\sigma_{DL}$}$$

where $w_0 = (c^n(i), c^n(k))\sigma_{DL}$.

A complex example of $\sigma_{DL}$-unification is provided by the labels

$$(d_2^s,(D_2,(W_1,(D_1,w_1^i)))) \qquad\qquad (S_1,(W_2,(s_2^i,w_1^i)))$$

$$((d_2^s,w_0),(S_1,(W_2,w_0))\sigma_{DL}$$

given

$$(d_2^s,S_1)\sigma^\# = (d_2^s,W_2)\sigma^+ = d_2^s$$

and

$$w_0 = ((D_2,(W_1,w_0')),(s_2^i,w_0'))\sigma_{DL}\ ,$$

since

$$(D_2,s_2^i)\sigma^\# = (W_1,s_2^i)\sigma^+ = s_2^i$$

and finally

$$w_0' = ((D_1,w_1^i),w_1^i)\sigma_{DL}\ .$$

### 3.3 *Labels, Unifications and Legal Reasoning*

What does a possible world denote? We suggest that a possible and plausible answer could be that a possible world of a given type represents an actual fact, and another type of possible world denotes laws. The unifications tell us when two labels are "matchable". If they are, we can compare whatever holds in the worlds they denote; therefore we can decide, analytically, whether a given fact is a violation of a law.

Obviously, according to our philosophical point of view, the formalization of norms will behave in different ways. We believe that our label manipulation could help to examine a few ideas about norms. Let us examine the basic cases of unifications

Case 1. A variable and a constant;

Case 2. Two constants;

Case 3. Two variables.

Roughly, cases 1, 2 and 3 correspond respectively to:

- Distribution axiom $\Box(A \to B) \to (\Box A \to \Box B)$;
- Necessitation rule $\dfrac{A}{\Box A}$;
- Kant's axiom $\Box A \to \Diamond A$.

Combining cases 1, 2 and 3 we can obtain different philosophical positions concerning norms.

Case 1 implies that norms express generic "situations" and we have to detect whether a given "situation" falls into the category of the generic one.

Case 2 implies that each norm expresses a given situation and we have to detect whether a given situation is the same as that of the norm, so each situation should have its specific norm.

Case 3 (idealization) implies the completeness of a normative system in a weak sense. So each instance of a situation should be determined by the norms. If there is a gap in a normative system, this condition states that norms themselves should give tools to fill the gap.

Almost every positive legal system has some mechanism to fulfil the requirement of the last case. For example, in Italian Law, article 12, 2° comma of the "*Preleggi*" prescribes analogical reasoning.

### 3.4 *Inference Rules*

We shall classify our inference rules in two main categories: structural rules and operational rules; the operational rules describe the meaning of the various operators and connectives involved (see D'Agostino and Mondadori 1994 for further explanations), whereas structural rules describe semantic properties holding in the model for the logic we are concerned with. Moreover it is possible to have other non-standard connectives and operators for which we can state their appropriate inference rules using labels (see D'Agostino and Gabbay 1994). The rules for the connectives are stated as follows[2]:

$$\frac{\alpha, j}{\alpha_1, j} \qquad \frac{\alpha, j}{\alpha_2, j}$$

$$\frac{\begin{array}{c}\beta, j\\ \beta_2^C, k\end{array}}{\beta_1, (j,k)\sigma_{DL}}\ (j,k)\sigma_{DL} \qquad \frac{\begin{array}{c}\beta, j\\ \beta_1^C, k\end{array}}{\beta_2, (j,k)\sigma_{DL}}\ (j,k)\sigma_{DL}$$

For the modal-like operators we have

$$\frac{T\mathsf{N}_D A, j}{TA, (W_n, j)} \qquad \frac{\nu_{\{i,s\}}A, j}{\nu_0, (\{D,S\}_n, j)}\ \{D,S\}_n \text{ new}$$

and

$$\frac{F\mathsf{N}_D A, i}{F\mathsf{O}^i A \wedge \mathsf{O}^s A, i} \qquad \frac{\pi_{\{i,s\}}, i}{\pi_0, (\{d,s\}_n, i)}\ \{d,s\}_n \text{ new}$$

The "standard" structural inference rules, respectively the principle of bivalence $(PB)$ and the principle of not contradiction $(PNC)$, are:

$$\frac{}{X, j \qquad X^C, j}\ h(j) \in \Phi_C \qquad \frac{\begin{array}{c}X, j\\ X^C, k\end{array}}{\times (j,k)\sigma_{DL}}\ (j,k)\sigma_{DL}$$

Here the $\alpha$-rules are just the familiar linear branch-expansion rules of the tableau method, while the $\beta$-rules correspond to such common natural inference patterns as *modus ponens*, *modus tollens*, etc. ($i, k, m$ stand for arbitrary labels). The rules for the modal operators are as usual. "New" in the proviso for the $\nu_{\{i,s\}}$- and $\pi_{\{i,s\}}$-rule means: $\{D,S\}_n$, $\{d,s\}_n$ must not have occurred in any label yet used. Notice that in all inferences via an $\alpha$-rule the label of the premise carries over unchanged to the conclusion, and in all inferences via a $\beta$-rule the labels of the premises must be $\sigma_{DL}$-unifiable, so that the conclusion inherits their unification. $PB$ (the "Principle of Bivalence") represents the ($LS$-version of the) semantic counterpart of the cut rule of the sequent calculus (intuitive meaning: a formula $A$ is either true or false in any *given* world, whence the requirement that $i$ should be restricted). $PNC$ (the "Principle of Non-Contradiction") corresponds to the familiar branch-closure rule of the tableau method, saying that from the occurrence of a pair of $LS$-formulas $X, i$, $X^C, k$ such that $(i,k)\sigma_{DL}$ (let us call them *$\sigma_{DL}$-complementary*) on a branch we may infer the closure ("$\times$") of the branch. The $(i,k)\sigma_{DL}$ in the "conclusion" of $PNC$ means that the contradiction holds "in the same world".

The peculiar structural inference rules of $DL$, the rules which represent the conditions of the model, are:

$$\frac{\begin{array}{c} X,(D,j) \\ X,(S,k) \end{array}}{X,(W_n,(j,k)\sigma_{DL})}\ (j,k)\sigma_{DL}$$

which states that a property holds universally. The main purpose of this rule is to ensure reflexivity with respect to $j$ and $k$, i.e, each world is either an ideal or a subideal version of itself; in fact a general property of labels and unifications states that

$$(j,k)\sigma_{DL} \Rightarrow ((j,k)\sigma_{DL},j)\sigma_{DL} \text{ and } ((j,k)\sigma_{DL},k)\sigma_{DL}$$

The next rule, $RR$ (Reflexivity Rule) tells us when a world is an ideal or subideal version of itself.

$$\frac{\begin{array}{c} \nu_{\{i,s\}},j \\ \nu_0^C,k \end{array}}{\begin{array}{c} \nu_{\{i,s\}},m^r \\ \nu_0^C,m^r \end{array}}\ m=(j,k)\sigma_{DL}$$

where

$$\begin{array}{ll} i^r = i^s & \text{if } \nu_{\{i,s\}} = T\mathsf{O}^i A\,(F\mathsf{P}^i A) \\ i^r = i^d & \text{if } \nu_{\{i,s\}} = T\mathsf{O}^s A\,(F\mathsf{P}^s A) \end{array}$$

and

$$i^x = i : h(i) \in \Phi^x, (x \in \{d,s\})$$

Obviously each $\Phi_X^r \subseteq \Phi_X$. We shall call labels of the form $i^x$, $(x \in \{d,s\})$ $x$-reflexive labels.

Besides the usual closure rule ($PNC$) and the principle of bivalence ($PB$) we introduce the following rules $LPNC$ and $LPB$

$$\frac{\begin{array}{c} j \in \Phi^s \\ j \in \Phi^d \end{array}}{\times} \qquad \frac{}{X,j^i \qquad K,j^s}$$

stating, respectively, that no world can be at the same time an ideal and a subideal version of itself and that each worlds is either an ideal or a subideal version of itself.

## 3.5 *Proof search*

] Let $\Gamma = \{X_1, \ldots, X_m\}$ be a set of $S$-formulas. Then $\mathcal{T}$ is a *KEM-tree for* $\Gamma$ if there exists a finite sequence $(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n)$ such that (i) $\mathcal{T}_1$ is a 1-branch tree consisting of $\{X_1, i \ldots, X_m, i\}$, where $i$ is an arbitrary constant label; (ii) $\mathcal{T}_n = \mathcal{T}$, and (iii) for each $i < n, \mathcal{T}_{i+1}$ results from $\mathcal{T}_i$ by an application of a rule of $KEM$. A branch $\tau$ of a $KEM$-tree $\mathcal{T}$ of $LS$-formulas is said to be $\sigma_{DL}$-$closed$ if it ends with an application of $PNC$, open otherwise. As usual with tableau methods, a set $\Gamma$ of formulas is checked for consistency by constructing a $KEM$-tree for $\Gamma$. It is worth noting that each $KEM$-tree is a (class of) Hintikka's model(s) where the labels denote worlds (i.e., Hintikka's modal sets), and the unifications behave according to the conditions placed on the appropriate accessibility relations. Moreover we say that a formula $A$ is a *KEM-consequence of a set of formulas* $\Gamma$ if $A$ occurs in all the open branches of a $KEM$-tree for $\Gamma$. We now describe a systematic procedure for $KEM$. First we define the following notions.

Given a branch $\tau$ of a $KEM$-tree, we shall call an $LS$-formula $X, i$ *E-analysed in* $\tau$ if either (i) $X$ is of type $\alpha$ and both $\alpha_1, i$ and $\alpha_2, i$ occur in $\tau$; or (ii) $X$ is of type $\beta$ and one of the following conditions is satisfied: (a) if $\beta_1^C, k$ occurs in $\tau$ and $(i,k)\sigma_{DL}$, then also $\beta_2, (i,k)\sigma_{DL}$ occurs in $\tau$; or (b) if $\beta_2^C, k$ occurs in $\tau$ and $(i,k)\sigma_{DL}$, then also $\beta_1, (i,k)\sigma_{DL}$ occurs in $\tau$; or (iii) $X$ is of type $\nu_i$ and $\nu_0, (m,i)$ occurs in $\tau$ for some $m \in \Phi_V$ not previously occurring in $\tau$; or (iv) $X$ is of type $\pi_i$ and $\pi_0, (m,i)$ occurs in $\tau$ for some $m \in \Phi_C$ not previously occurring in $\tau$.

We shall call a branch $\tau$ of a $KEM$-tree $E$-$completed$ if every $LS$-formula in it is $E$-analysed and it contains no complementary formulas which are not $\sigma_{DL}$-complementary. We shall call a branch $\tau$ of a $KEM$-tree $completed$ if it is $E$-completed and all the $LS$-formulas of type $\beta$ in it either are analysed or cannot be analysed. We shall call a $KEM$-tree $completed$ if every branch is completed.

The following procedure starts from the 1-branch, 1-node tree consisting of $\{X_1, i_1, \ldots, X_m, i_m\}$ and applies the rules of $KEM$ until the resulting $KEM$-tree is either closed or completed.

We shall say that a formula $A$ is a theorem of $DL$ when a closed $KEM$-tree for $FA, w_1$ exists.

At each stage of proof search (i) we choose an open non completed branch $\tau$. If $\tau$ is not $E$-completed, then (ii) we apply the 1-premise rules until $\tau$ becomes $E$-completed. If the resulting branch $\tau'$ is neither closed nor completed, then (iii) we apply the 2-premise rules until $\tau$ becomes $E$-completed. If the resulting branch $\tau'$ is neither closed nor completed, then (iv) we choose an $LS$-formula of type $\beta$ which is not yet analysed in the branch and apply $PB$ so that the resulting $LS$-formulas are $\beta_1, i'$ and $\beta_1^C, i'$ (or, equivalently $\beta_2, i'$ and $\beta_2^C, i'$), where $i = i'$ if $i$ is restricted, otherwise $i'$ is obtained from $i$ by instantiating $h(i)$ to a constant not occurring in $i$; (v) ("Modal $PB$") if the branch is not $E$-completed nor closed, because of complementary formulas which are not $\sigma_{DL}$-complementary, then we have to see whether a restricted label unifying with both the labels of the complementary formulas occurs previously in the branch; if such a label exists, or can be built using already existing labels and the unification rules, then the branch is closed; (vi) ("Label $PB$") if the branch is not $E$-completed nor closed, because of complementary formulas which are not $\sigma_{DL}$-complementary and the heads of their labels, $j, k$, are respectively in $\Phi_D$ and $\Phi_S$, then we have to see whether there exists a restricted non reflexive label, that, when it is $d$-reflexive, unifies with $j$ and when it is $s$-reflexive unifies with $k$; if such a label exists, or can be built using already existing labels and unification rules, then the branch is closed; (vii) we repeat the procedure in each branch generated by $PB$.

The above procedure is based on a (deterministic) procedure working for *canonical* $KEM$-trees. A $KEM$-tree is said to be canonical if it is generated by applying the rules of $KEM$ in the following fixed order: first the $\alpha$-, $\nu_{\{i,s\}}$- and $\pi_{\{i,s\}}$-rule, then the $\beta$-rule and $PNC$, and finally $PB$. Two interesting properties of canonical $KEM$-trees are (i) that a canonical $KEM$-tree always terminates, since for each formula there are a finite number of subformulas and the number of labels which can occur in the $KEM$-tree for a formula $A$ (of $L$) is limited by the number of modal operators belonging to $A$, and (ii) that for each closed $KEM$-tree a closed canonical $KEM$-tree exists. Proofs of termination and completeness for canonical $KEDL$-trees follow by obvious modifications of the proofs given in Governatori 1995.

*Remark* 2. We distinguish between $DL$-theories, obtained by means of configurations of possible worlds, and $DL$ obtained by means of the above inference rules and unifications. It is worth noting that labels allow us not only to manipulate formulas in deductions but also worlds, which turns out to be very important when dealing with theories (For a similar approach see Russo 1996).

The following are example proofs of theorems of $DL$.

$$
\begin{array}{lr}
1.\ F(\mathsf{O}^s A \wedge \neg A) \rightarrow \mathsf{P}^s \neg A & w_1 \\
2.\ T\mathsf{O}^s A \wedge \neg A & w_1 \\
3.\ F\mathsf{P}^s \neg A & w_1 \\
4.\ T\mathsf{O}^s A & w_1 \\
5.\ FA & w_1 \\
6.\ T\mathsf{O}^s A & w_1^s \\
7.\ FA & w_1^s \\
8.\ TA & S_1, w_1^s \\
9.\quad \times &
\end{array}
$$

The steps leading to the nodes (1)-(5) are straightforward. The nodes (6)-(7) come from the application of the reflexivity rule since the world denoted by $w_1$ is a sub-ideal version of itself. Closure follows immediately from (7) and (8), which are $\sigma_{DL}$-complementary (their labels $\sigma_{DL}$-unify because of $(S_1, w_1^s)\sigma^{\#}$).

| | |
|---|---|
| 1. $F\mathsf{O}^i A \rightarrow (\mathsf{O}^s B \rightarrow (\neg A \rightarrow B))$ | $w_1$ |
| 2. $T\mathsf{O}^i A$ | $w_1$ |
| 3. $F\mathsf{O}^s B \rightarrow (\neg A \rightarrow B)$ | $w_1$ |
| 4. $T\mathsf{O}^s B$ | $w_1$ |
| 5. $F\neg A \rightarrow B$ | $w_1$ |
| 6. $FA$ | $w_1$ |
| 7. $FB$ | $w_1$ |
| 8. | $w_1^s$ |
| 9. | $w_1^i$ |
| 10. | $\times$ |

Here the steps (8) and (9) are obtained, respectively, from (2), (6) and (4), (7) by $RR$, and the closure follows from an application of $LPNC$.

| | |
|---|---|
| 1. $F(\mathsf{O}^i(A \wedge B) \wedge \mathsf{O}^s(C \wedge D)) \rightarrow (A \vee C)$ | $w_1$ |
| 2. $T\mathsf{O}^i(A \wedge B) \wedge \mathsf{O}^s(C \wedge D)$ | $w_1$ |
| 3. $FA \vee C$ | $w_1$ |
| 4. $T\mathsf{O}^i(A \wedge B)$ | $w_1$ |
| 5. $T\mathsf{O}^s(C \wedge D)$ | $w_1$ |
| 6. $FA$ | $w_1$ |
| 7. $FC$ | $w_1$ |
| 8. $TA \wedge B$ | $D_1, w_1$ |
| 9. $TC \wedge D$ | $S_1.w_1$ |
| 10. $TA$ | $D_1, w_1$ |
| 11. $TB$ | $D_1, w_1$ |
| 12. $TC$ | $S_1.w_1$ |
| 13. $TD$ | $S_1.w_1$ |
| 14. $T \quad w_1^i$ | 15. $F \quad w_1^s$ |
| 16. $\times$ | 17. $\times$ |

In the left branch, closure follows from $TA, (D_1, w_1)$, $FA, w_1$ and $w_1^i$, after we have assumed, through the label version of $PB$, that $w_1$ is an ideal version of itself, i.e., $w_1^i$; we replace, with respect to the left branch, all the occurrences of $w_1$ with $w_1^i$ thus obtaining $D_1, w_1^i$ and $w_1^i$ which $\sigma_{DL}$-unify; on the other hand, in the right branch we have $TC, (S_1, w_1)$, $FC, w_1$ and $w_1^s$, and we can repeat the same procedure as for the left side.

## 4 Final Remarks

Although a satisfactory Logical System for Law is far from being realized, we believe that the approach we have presented may offer a few steps in the right direction. In fact, the label tool we have developed is flexible enough to cope with several types of modal-like notions of obligatoriness at the same time, and to study their mutual relationships through unifications. It often happens that, in a legal system, laws prescribe opposite possibilities for the same fact according to "relevant" pieces of evidence; for example, some legal system could prescribe a murder to be punished unless he/she killed in self-defence. Logically this scenario is contradictory because, in the case of self-defence, both punishment and not punishment are implied; however it is possible to solve this problem as soon as some refinement is assumed (on this point see Artosi, Governatori and Sartor 1996). Moreover, as we have already seen, different traits of legal reasoning might involve different kinds of logics (even with different connectives and operators); the resulting overall logic can be embedded in the so-called fibred semantics (logic) framework (Gabbay 1994; 1996), but the label formalism here presented can be extended, straightforwardly, to deal with it.

The preceding discussion was thus mainly aimed at showing the potential scope of application of the method. In effect, we believe that the method we proposed to determine the ideal/subideal status of world nicely exploits the computational and proof-theoretical advantages offered by the modal theorem proving system $KEM$. As we have argued elsewhere, this system enjoys most of the features a suitable proof search system for modal (and in general non-classical) logics should have. In contrast with (both clausal and non-clausal) resolution methods, and in general "translation-based" methods (Ohlbach 1991), it works for the full modal language (thus avoiding any preprocessing of the input formulas), and it is flexible enough to be extended to cover any setting having a Kripke-model based semantics (this is clearly shown by our treatment of Jones and Pörn logic $DL$ where the rules specific for such a logic should take care not only of the propositional and modal part but also of the structure of the labels and the relationship between labels and formulas; for example we added another closure rule $\frac{i \in \Phi^i, i \in \Phi^s}{\times}$ which states that no world can be at the same time an ideal and a sub-ideal version of itself; this result is achieved by determining when a deontic world is ideally (sub-ideally) reflexive ($i^r$) by means of another peculiar inference rule, and finally the principle of bivalence for labels). From this perspective our method is similar to the natural deduction proof method proposed by Russo (1996). Nevertheless, it has several advantages over most tableau/sequent based theorem proving methods: being based on D'Agostino and Mondadori's classical proof system $KE$, it eliminates the typical redundancy of the standard cut-free methods and, thanks to its label unification scheme, it offers a simple and efficient solution to the permutation problem which notoriously arises at the level of the usual tableau-sequent rules for the modal operators (Fitting 1988). However, unlike, e.g., Wallen's (1990) connection method, it uses a natural and easily implementable style of proof construction, and so it appears to provide an adequate basis for combining both efficiency and naturalness.

## Notes

1. Herein with $\Box$ we mean any modality which acts as $\Box$, i.e., $\mathsf{O}^i$ and $\mathsf{O}^s$ ; and with $\Diamond$ any modality which acts as $\Diamond$, i.e., $\mathsf{P}^i$ and $\mathsf{P}^s$.
2. The following formulation uses a generalized $\alpha, \beta, \nu_{\{i,s\}}, \pi_{\{i,s\}}$ form of Smullyan-Fitting $\alpha, \beta, \nu, \pi$ unifying notation, see Fitting 1983.

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