Chapter 7

# FIBRED MODAL TABLEAUX

Dov M. Gabbay
*Department of Computer Science*
*King's College, London*
dg@dcs.kcl.ac.uk

Guido Governatori
*School of Computing and Information Technology*
*Griffith University, Brisbane*
G.Governatori@cit.gu.edu.au

**Abstract** We describe a general and uniform tableau methodology for multi-modal logics arising from Gabbay's methodology of fibring and Governatori's labelled tableau system KEM.

## 1. INTRODUCTION

Gabbay [11, 14] presented a general methodology for combining logical systems called fibring. Beckert and Gabbay [5] have studied under which conditions proof systems are well-suited for fibring, and they have argued that labelled tableaux are calculi of this kind. Moreover D'Agostino et al. [7, 8] have shown how to extend classical proof system KE [9], a combination of tableaux and natural deduction inference rules which allows for a restricted ('analytic') use of the cut rule, to deal with fibred logics, in particular for a combination of modal and substructural logics and the combination of different fragment of substructural implications. In [1, 18] a labelled extension of KE, called KEM, has been proven to be able to cope with a wide variety of (normal) modal logics, and in [3, 16, 17, 18] it has been showed how such a system can be extended to deal with multi-modal logics. In the present work we show how to adapt KEM in order to obtain a general and uniform tableau-like proof method for the fibred (dovetailed) combination of modal logics. In

D. Basin, M. D'Agostino, D. Gabbay. S. Matthews and L. Viganó (eds)
*Labelled Deduction*, pp. 163–193.

particular we show how the label formalism used in KEM matches Gabbay's semantic based fibring for (multi-) modal logics. The resulting system enjoys several interesting properties: it is modular, in the sense that the proof system for each logic is developed on its own and it is reused in the combination; it is uniform, in the sense that each system has its own individual features, but the framework remains constant among the combined systems; it seems to be flexible enough to deal with other logics, as well as their combinations; finally, although we use some logical machinery in describing the system, our presentation is natural and the idea behind it is very simple and easy to grasp.

Let $L_1$ and $L_2$ be two modal logics for which a tableau system exists. We start a tableau for a formula $A$ of $L_1$, which has formulas of $L_2$ embedded in it. In the course of the proof for $A$, when we have to process a formula of $L_2$ we begin a new proof for it in the appropriate system. It turns out that we do not really need to start a separate tableau, but we graft it into the original one. The labels will have a structure which will store nicely all the information about the grafting operation. This is possible because the key feature of KEM, besides its being based neither on resolution nor on standard sequent/tableau inference techniques, is that it generates models and checks them using a label scheme during the bookkeeping for the fibred model. The mechanism which KEM uses in manipulating labels is close to semantic fibring (dovetailing).

In this essay we shall stick ourselves with KEM based proof method, nevertheless the label formalism, and the combining methodology, can be used with almost whatever proof system allowing: 1) atomic proofs, and 2) composition of deductions.

The paper is organized as follows: in section 2. we shall introduce two basic methods for combining logical systems, namely fibring (2.1) and dovetailing (2.2); in section 4. we recall the basic tableau system KEM for modal logics; we shall relate the techniques from the previous sections in order to provide a general and uniform tableau methodology for multi-modal logics.

## 2. COMBINING MODAL LOGICS

The methodology of fibring allows us to combine arbitrary logical systems to form a new system in a uniform way "fibring" their models (for detailed expositions see [11, 12, 13, 14]). The main idea of fibring is very simple, and provides a new concept of possible world semantics. Let us suppose we want to combine two modalities $\Box_1$ and $\Box_2$ characterised, respectively, by the classes of models $\mathcal{K}_1$ and $\mathcal{K}_2$. We know how to evaluate $\Box_1 A$ in $\mathcal{K}_1$, $\Box_2 A$ in $\mathcal{K}_2$ and propositional formulas in both. All we need is a method for evaluating $\Box_1$ (resp. $\Box_2$) w.r.t. $\mathcal{K}_2$ (resp. $\mathcal{K}_1$). Each time we have to evaluate a formula $\mathcal{A}$ of the form $\Box_2 A$ in a world in a model of $\mathcal{K}_1$ we associate, via the fibring function **F**, to the world a model in $\mathcal{K}_2$, or its actual world, where we calculate the truth value of the formula. Formally

$$w \models_{\mathbf{m}\in\mathcal{K}_1} \Box_2 A \iff \mathbf{F_m}(w) \models_{\mathbf{m}'\in\mathcal{K}_2} \Box_2 A$$

$\mathcal{A}$ holds in $w$ iff it holds in the model associated to $w$ through the fibring function $\mathbf{F}$ (i.e., $\mathbf{m}'$); more precisely $\mathcal{A}$ holds in the actual world (i.e., $\mathbf{F_m}(w)$) of $\mathbf{m}'$. But now we are in an appropriate model for evaluating $\mathcal{A}$.

In the next two sections we define two ways, fibring and dovetailing, in which semantics can be combined. We first explain the concepts by taking a simple example. Suppose we want to combine two modal logics $L_1$ and $L_2$. Let $\mathcal{K}_1, \mathcal{K}_2$ be the respective Kripke semantics of the logic. Let $\mathbf{m}$ be a model in $\mathcal{K}_1$ and let $t$ be a possible world of $\mathbf{m}$. The semantic construction which combines the logics associates to $t$ a model $\mathbf{n}$ in $\mathcal{K}_2$. The different methodologies of combination differ on the kind of model $\mathbf{n}$ we use. For *fibring* logics, we require that $\mathbf{n}$ be any model in $\mathcal{K}_2$. For *dovetailing*, a special case of fibring, we require that $\mathbf{n}$ be a model of $\mathcal{K}_2$ such that for any *atomic* $p$

$$t \models p \text{ iff } \mathbf{n} \models p$$

(i.e., the fibred model must agree with the values $t$ gives to atoms).

First of all we have to introduce the language of the combined logic. Let $I$ be a set of indices, and let $L_i, i \in I$ be modal logics in the respective language $\mathbb{L}_i$, with $\Box_i, i \in I$, respectively. The expressions $E^i$ of the language $\mathbb{L}_i$ are built up using $\mathbb{L}_i$-constructors (connectives and operators) from a set of atomic units $p^i$. We schematically write $E^i(p^i_1, \ldots, p^i_n)$ to indicate that $E^i$ is built up from the atoms $p^i_1, \ldots, p^i_n \in p^i$.

**Definition 1** *The fibred language $\mathbb{L}_{(x_1,\ldots,x_n)}$ is defined as follows:*

- *Let $\mathbb{L}_{(i)}$ be $\mathbb{L}_i$, $i \in I$;*
- *Let $\bar{y}$ be $(j, y_1, \ldots, y_k)$, $j, y_1, \ldots, y_k \in I$ and $i \neq j$. Let $\mathbb{L}_{(i)*\bar{y}}$ be the family of all expressions of the form $\alpha \in \mathbb{L}_{(y_1,\ldots,y_k)}$ or $\alpha = E^i(p_1/A_1, \ldots, p_n/A_n)$ where $E^i(p_1, \ldots, p_n) \in \mathbb{L}_i$ and $A_1, \ldots, A_n$ are in $\mathbb{L}_{\bar{y}}$, and $p_k/A_k$ indicates the substitution of $A_k$ to $P_k$ in $E^i$;*
- $\mathbb{L}_I = \bigcup_{\bar{y}} \mathbb{L}_{\bar{y}}$.

To clarify this notion let us consider two modal languages $\mathbb{L}_1$ and $\mathbb{L}_2$. The fibred language $\mathbb{L}_{(2,1,y_1,\ldots,y_k)}$ is the set of all expressions with outer constructor from $\mathbb{L}_2$ and with no more than $k+2$ nested alternation of constructors from $\mathbb{L}_1$ and $\mathbb{L}_2$.

## 2.1 FIBRING MODAL LOGICS

Let $\mathcal{K}_i$ be a class of models $\{\mathbf{m}^i_1, \mathbf{m}^i_2, \ldots\}$ for which $L_i$ is complete. Each model $\mathbf{m}^i_n$ has the form $(S, R, a, h)$ where $S$ is the set of possible worlds,

$a \in S$ is the actual world and $R \subseteq S^2$ is the accessibility relation. $h$ is the assignment function, a binary function, giving a value $v(t,p) \in \{1,0\}$ for any $t \in S$ and atomic $p$. The actual world $a$ plays a role in the semantic evaluation in the model, in so far as satisfaction in the model is defined as satisfaction at $a$. We can assume that the models satisfy the following condition:

$$S = \{x \mid \exists n \;\; aR^n x\} \ .$$

This assumption does not affect satisfaction in models because points not accessible from $a$ by any power $R^n$ of $R$ do not affect truth values at $a$. Moreover we assume that all sets of possible worlds in any $\mathcal{K}_i$ are all pairwise disjoint, and that there are infinitely many isomorphic (but disjoint) copies of each model in $\mathcal{K}_i$. We use the notation $\mathbf{m}$ for a model and present it as $\mathbf{m} = (S^{\mathbf{m}}, R^{\mathbf{m}}, a^{\mathbf{m}}, h^{\mathbf{m}})$ and write $\mathbf{m} \in \mathcal{K}_i$, when the model $\mathbf{m}$ is in the semantics $\mathcal{K}_i$. Thus our assumption boils down to $\mathbf{m} \neq \mathbf{n} \Rightarrow S^{\mathbf{m}} \cap S^{\mathbf{n}} = \varnothing$. In fact a model can be identified by its actual world, i.e., $\mathbf{m} = \mathbf{n}$ iff $a^{\mathbf{m}} = a^{\mathbf{n}}$.

**Definition 2** *A fibred model is a structure*

$$(W, W_{i,i\in I}, W_a, R, w_0, h, \mathbf{F})$$

*where* $W = \bigcup_{\mathbf{m}\in\cup_i\mathcal{K}_i} S^{\mathbf{m}}$*;* $W_i = \{a^{\mathbf{m}} \mid \mathbf{m} \in \mathcal{K}_i\}$*;* $W_a = \bigcup_i W_i$*;* $R = \bigcup_{\mathbf{m}\in\cup_i\mathcal{K}_i} R^{\mathbf{m}}$*;* $w_0 \in W_a$ *is the actual world;* $h(t,p) = h^{\mathbf{m}}(t,p)$*, for the unique* $\mathbf{m}$ *such that* $t \in S^{\mathbf{m}}$*;* $\mathbf{F} : I \times W \mapsto W_i$*, is the fibring function. The fibring function* $\mathbf{F}$ *is a function giving for each* $i \in I$ *and each* $w \in W$ *another point (actual world) in* $W_i$ *as follows:*

$$\mathbf{F}_i(w) = \begin{cases} w & \text{if } w \in S^{\mathbf{m}} \text{ and } \mathbf{m} \in \mathcal{K}_i \\ \text{a value in } W_i, & \text{otherwise} \end{cases}$$

*such that if* $x \neq y$*, then* $\mathbf{F}_i(x) \neq \mathbf{F}_i(y)$*. Satisfaction is defined as follows with the usual truth tables for boolean connectives:*

$$t \vDash p \quad \text{iff} \quad v(t,p) = 1$$

$$t \vDash \Box_i A \quad \text{iff} \quad \begin{cases} t \in \mathbf{m}^i \text{ and } \forall s(tRs \to s \vDash A) \\ t \in \mathbf{m}^j,\ i \neq j \text{ and } \mathbf{F}_i(t) \vDash \Box_i A \end{cases}$$

*We say the model satisfies* $A$ *iff* $w_0 \vDash A$*.*

**Theorem 3** (Completeness theorem for the fibred logic $L_I^F$) *Let* $L_i$*,* $i \in I$ *be modal logics in the respective language* $\mathbb{L}_i$ *with classes of structures* $\mathcal{K}_i$ *and set of theorems* $\mathbf{T}_i$ *(i.e.,* $\mathbf{T}_i = \{A \text{ of } \mathbb{L}_i \mid A \text{ is valid in all } \mathcal{K}_i \text{ models}\}$*). Let* $\mathbf{T}_I^F$ *be the following set of wffs of* $L_I^F$*.*

1. $\mathbf{T}_i \subseteq \mathbf{T}_I^F$*, for every* $i \in I$*.*

1a. *If $A$ is a Boolean combination of atoms and $\alpha_n^{\bar{y}}$ is in the $\bar{y}$-th fibred language, then $A \to \alpha_n^{\bar{y}} \in \mathbf{T}_{\bar{y}}$ implies $A \to \bigvee_n \alpha_n^{\bar{y}} \in \mathbf{T}_I^F$, where every atom in $\alpha_n^{\bar{y}}$ is in the scope of a modality.*

1b. *If $A(x_k) \in \mathbf{T}_i$ then $A(x_k/\Box_j \alpha_k) \in \mathbf{T}_I^F$, for any $\Box_j \alpha_k \in \mathbb{L}_j, j \in I$.*

2. **Modal Fibring Rule:**
*If $\Box_i$ is the modality of $L_i$ and $\Box_j$ of $L_j$, where $i, j$ are arbitrary with $i \neq j$ and*

$$C = \bigwedge_{k=1}^{n} \Box_i A_k \to \bigvee_{k=1}^{m} \Box_i B_k \in \mathbf{T}_I^F$$

*then for all $d \in \mathbb{N}$, $\Box_j^d C \in \mathbf{T}_I^F$.*[1]

3. *$\mathbf{T}_I^F$ is the smallest set closed under 1, 2, modus ponens and substitution.*

*Then $\mathbf{T}_I^F$ is the set of all wffs of $L_I^F$ valid in all the fibred structures of $L_I^F$.*

**Proof** See [11, 14]. Notice that [11] deals with the case that the logics $L_i, i \in I$ have no atoms in common. ■

## 2.2 DOVETAILING MODAL LOGICS

Dovetailing arises in many applications where the fibred model at world $t$ has the world $t$ itself as its actual world. The notions of dovetailing results from that of fibring when for all $i \in I$ and for all atomic $p$

$$v(t, p) = v(\mathbf{F}_i(t), p) \; .$$

In such a case we can identify the actual world of the model fibred at $t$, $\mathbf{F}_i(t)$, with $t$. The fibring function $\mathbf{F}$ is no longer needed, since we identified $t$ with $\mathbf{F}_i(t)$.

Let $L_i, i \in I$ be modal logics with $\mathcal{K}_i$ the class of models for $L_i$. Let $L_I^D$ (the dovetailing combination of $L_i, i \in I$) be defined semantically through the class of all (dovetailed) models of the form $(W, R, a, v)$, where $W$ is a set of worlds, $a \in W$, $v$ is an assignment as before, and for each $i \in I, R(i) \subseteq W \times W$. We require that for each $i$ $(W, R(i), a, v)$ is a model in $\mathcal{K}_i$. We further require the following: Let $t \in W$ be such that there exist $n_1, \ldots, n_k$ and $i_1, \ldots, i_k$ such that $aR^{n_1}(i_1) \circ R^{n_2}(i_2) \ldots \circ R^{n_k}(i_k)t$ holds.

We define the notion of $w \vDash A$ by induction.

---

[1] As a notation we use $\Box_i^0 A$ for $A$ and $\Box_i^{n+1} A$ for $\Box_i \Box_i^n A$.

- $w \vDash p$ iff $v(w, p) = 1$ for $p$ atomic.
- $w \vDash \Box_i A$ if for all $y \in W$, such that $wR(i)y$ we have $y \vDash A$.
- $\vDash A$ iff for all models and actual worlds $a \vDash A$.

**Theorem 4** (Completeness theorem for the dovetailed logic $L_I^D$) *Let $L_i, i \in I$ be modal logics with semantical classes of structures $\mathcal{K}_i$ and set of theorems $\mathbf{T}_i$. Let $\mathbf{T}_I^D$ be the following set of wffs of $L_I^D$.*

1. *$\mathbf{T}_i \subseteq \mathbf{T}_I^D$, for every $i \in I$.*
2. **Modal Dovetailing Rule:**
   *If $\Box_i$ is the modality of $L_i$ and $\Box_j$ that of $L_j$, where $i, j$ are arbitrary with $i \neq j$, and*

$$C = \bigwedge_{k=1}^{n} \Box_i A_k \wedge \bigwedge_{k=1}^{m} \Diamond_i \neg B_k \rightarrow \bigvee_{k=1}^{r} p_k \in \mathbf{T}_I^D ,$$

   *then for all $d \in \mathbb{N}$, $\Box_j^d C \in \mathbf{T}_I^D$. Where $p_k$ are atoms or their negations, and $p_1, \ldots, p_r$ list all the atoms or their negations appearing in any $A_k$ or $B_k, k = 1, 2, \ldots$.*
3. *$\mathbf{T}_I^D$ is the smallest set closed under 1, 2, modus ponens and substitution.*

*Then $\mathbf{T}_I^D$ is the set of all wffs of $L_I^D$ valid in all the dovetailed structures of $L_I^D$.*

**Proof** See [11]. ■

## 3. FROM FIBRING TO LABELLED TABLEAUX

Let us explain the fibring idea by looking at an example. This example will be treated in Section 4.3 below.

Consider two logics $L_1$ and $L_2$ with modalities $\{\Box_1, \Diamond_1\}$ and $\{\Box_2, \Diamond_2\}$ respectively. Assume $L_i$ is an extension of $\mathbf{K}$ for its modality. Consider:

$$A = \Box_1 \Diamond_2 p \rightarrow \Diamond_1 \Box_2 \Diamond_2 p$$

This formula is in the combined language. We know how to dovetail these two logics to form a logic for the combined language. We know that if $L_1$ and $L_2$ are complete for classes of Kripke model $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively, then we can fibre (dovetail) the models and get a class $\mathcal{K}$ of models of the form $\mathbf{m} = (S, R_1, R_2, w_0, v)$, with two accessibility relations. The class $\mathcal{K}$ is constructed from $\mathcal{K}_1$ and $\mathcal{K}_2$. It contains some specific models of the form above and completeness holds.

Let us see whether we can find an $\mathbf{m}$ which is a counter-model for $A$. Let us check what is needed to be in $\mathbf{m}$ and then we can check whether such an $\mathbf{m}$ can be in the class $\mathcal{K}$. We want to have

$$1.\ w_0 \vDash \Box_1\Diamond_2 p \qquad\qquad 2.\ w_0 \nvDash \Diamond_1\Box_2\Diamond_2 p$$

From (1) we get that at an arbitrary point $t$ such that $w_0R_1t$ we must have $t \vDash \Diamond_2 p$. Let $t = W_1^1$ be such an arbitrary point. We use capital letters to indicate that the point $t$ is *arbitrary* and we use the superscript '1' to indicate that $w_0R_1t$ holds (i.e. we are choosing $t$ because we are evaluating a modality of $L_1$). The subscript just counts that this is the first '$W^1$' we are using. Hence we have

$$W_1^1 \vDash \Diamond_2 p.$$

Now we are evaluating the second modality. Therefore there exists a specific point $s$ such that $W_1^1R_2s$ and $s \vDash p$. Again we use the suggestive notation $s = w_1^2$ and write $w_1^2 \vDash p$. The lower case $w$ indicates that $w_1^2$ is not arbitrary but specific and the superscript '2' indicates that we are dealing with an $L_2$ modality. Similarly, since

$$w_0 \nvDash \Diamond_1\Box_2\Diamond_2 p$$

we must have that for an arbitrary $x$ such that $w_0R_1x$ we have $x \nvDash \Box_2\Diamond_w p$. According to our agreed notation, we represent such an arbitrary $x$ by

$$W_2^1 \nvDash \Box_2\Diamond_2 p.$$

The superscript '1' indicates that this point arose because of an $L_1$ modality. The subscript '2' indicates that this is a second such point used so far and the capital $W$ (as opposed to a lower case $w$) indicates that it is an arbitrary point (accessible to $w_0$). We can continue, since

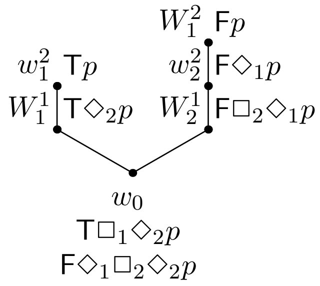
$$W_2^1 \nvDash \Box_2\Diamond_2 p$$

there exists a specific point $y$, $W_2^1R_2y$ such that $y \nvDash \Diamond_2 p$. Using our conventions, we represent this point by

$$w_2^2 \nvDash \Diamond_2 p.$$

We can continue and represent an arbitrary point $W_1^2$ such that $w_2^2R_2W_1^2$ and $W_1^2 \nvDash p$. We can represent what we got so far in a tree in Figure 7.1(a).

The $\mathsf{T}$, $\mathsf{F}$ indicate whether the formula is supposed to hold or not in the node. The uppercase/lowercase distinction tells us whether the point is arbitrary or not and the superscript tells us whether the point is $R_1$ or $R_2$ accessible to the




(a)

| Point in tree | Sequence Representation |
|---|---|
| $w_0$ | $w_0$ |
| $W^1_2$ | $(W^1_2, w_0)$ |
| $w^2_2$ | $(w^2_2, (W^1_2, w_0))$ |
| $W^2_1$ | $(W^2_1, (w^2_2, (W^1_1, w_0)))$ |
| $W^1_1$ | $(W^1_1, w_0)$ |
| $w^2_1$ | $(w^2_1, (W^1_1, w_0))$. |

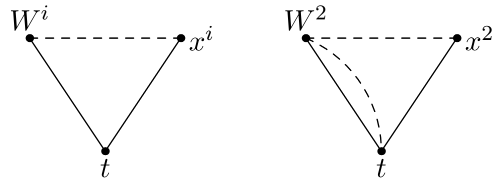
(b)

*Figure 7.1*

point below it. We are relying on the geometrical drawing of a tree to represent the situation. If we do not want to draw trees, we can represent each point as the *sequence* of points leading up to it. Thus we have the table in Figure 7.1(b):

We now address the question of whether the situation in Figure 7.1(a) does provide a counter-model for $A$. Of course, if both $\Box_1$ and $\Box_2$ are $\mathbf{K}$ modalities, then we can provide a counter-model for $A$, since in this case, the dovetailed semantics $\mathcal{K}$ contains any model $\mathbf{m}$ with arbitrary $v$, $R_1$ and $R_2$. We know this fact from the completeness theorem of dovetailed logic (Theorem 4). If we do not want to rely on semantics, we can ask what would be the corresponding condition on the trees obtained (as we see in the example of Figure 7.1(a)) that will tell us whether we can get a counter-model or not. In other words, is the tree 'closed' therefore there is no counter-model, i.e. $A$ is a theorem?

The following are the kind of rules we want:




(a) (b)

*Figure 7.2*

In the subtree of Figure 7.2(a), $W^i$ can unify with $x^i$. By repeated applications of the rule we try to unify points and get a contradiction. For example, if in Figure 7.2(a) we have $W^i$ T$p$ and $x^i$ F$p$ we get a contradiction.

Different logics will have different unification rules. For example, if the logic $L_2$ is a $\mathbf{T}$ modality, then in this case we use the rule depicted in Figure 7.2(b), where $W^2$ can unify with either $x^2$ or $t$.

A tableaux system for a dovetailed logic with $\{\Box_1, \Box_2\}$ would need the following components.

- Rules for building trees as we did in Figure 7.1(a) or using sequences as in Figure 7.1(b).

- Unification rules that unify labels in order to identify two labels; for example, one with $\mathsf{F}q$ and the other with $\mathsf{T}q$.

- Completeness theorem for the resulting tableaux system for the dovetailed logic.

The next sections do all that in detail. It is a messy business and is done best by a theorem prover. Work on functional translation of this kind has been done by Ohlbach [20], and optimisation have been proposed in [21] and by Gasquet [15].

## 4. LABELLED TABLEAUX FOR MODAL LOGICS

In [1, 18] a tableau-like proof system, called KEM, has been presented, and it has been proven to be able to cope with a wide variety of (normal) modal logics. KEM is based on D'Agostino and Mondadori's [9] classical proof system KE, a combination of tableau and natural deduction inference rules which allows for a restricted ("analytic") use of the cut rule. The key feature of KEM, besides its being based neither on resolution nor on standard sequent/tableau inference techniques, is that it generates models and checks them using a label scheme for bookkeeping fibred models. In [3, 16, 17, 18] it has been shown how this formalism can be extended to handle various systems of multi-modal logic with interaction axioms. The mechanism KEM uses in manipulating labels is close to semantic fibring (dovetailing).

### 4.1 LABEL FORMALISM

In this section we introduce the label formalism we shall use in the course of the chapter. KEM uses *Labelled Signed Formulas* (*LS*-formula for short), where an *LS*-formula is an expression of the form $SA : t$, where $A$ is a wff of the logic, $S$ (the truth sign) is in $\{\mathsf{T}, \mathsf{F}\}$, and $t$ is a label. Notice that the label we are referring in this section are not the labels of $I$. In the case of modal logic we have a type of labels corresponding to various modalities, and each set of atomic labels is partitioned into the set of variables and the set of constants. Formally

$$\Phi^{i\in I} = \Phi_C^{i\in I} \cup \Phi_V^{i\in I}$$

where $\Phi_C^{i\in I} = \{w_1^i, w_2^i, \dots\}$ and $\Phi_V^{i\in I} = \{W_1^i, W_2^i, \dots\}$. The set of constant world symbols and variable world symbols are respectively

$$\Phi_C = \bigcup_{i\in I} \Phi_C^{i\in I} \qquad \Phi_V = \bigcup_{i\in I} \Phi_V^{i\in I}$$

The set of labels $\Im$ is then defined inductively as follows:

$$\Im = \bigcup_{1\leq k} \Im_k \text{ where } \Im_k, k \in \mathbb{N} :$$

$$\Im_1 = \Phi_C \cup \Phi_V;$$
$$\Im_2 = \Im_1 \times \Phi_C;$$
$$\Im_{n+1} = \Im_1 \times \Im_n, (n > 1).$$

According to the above definition a label is either a (i) an element of the set $\Phi_C$, or (ii) an element of the set $\Phi_V$, or (iii) a path term $(s', s)$ where (iiia) $s' \in \Phi_C \cup \Phi_V$ and (iiib) $s \in \Phi_C$ or $s = (t', t)$ where $(t', t)$ is a label. From now on we shall use $t, s, r, \dots$ to denote arbitrary labels.

As an intuitive explanation, we may think of a label $t \in \Phi_C$ as denoting a world (a *given* one), and a label $t \in \Phi_V$ as denoting a set of worlds (*any* world) in some Kripke model. A label $s = (t', t)$ may be viewed as representing a path from $t$ to a (set of) world(s) $t'$ accessible from $t$ (i.e., from the world(s) denoted by $t$).

**Definition 5** *For any label $t = (s', s)$ we shall call $s'$ the* head *of $t$, $s$ the* body *of $t$, and denote them by $h(t)$ and $b(t)$ respectively.*

Notice that these notions are recursive (they correspond to projection functions): if $b(t)$ denotes the body of $t$, then $b(b(t))$ will denote the body of $b(t)$, $b(b(b(t)))$ will denote the body of $b(b(t))$; and so on.

**Definition 6** *We call each of $b(t)$, $b(b(t))$, etc., a* segment *of $t$. Let $b^m(t)$, be any segment of $t$ (obviously, by definition every segment $s(t)$ of a label $t$ is a label); then $h(b(t))$ will denote the head of $b^m(t)$. We shall call a label $t$* restricted *if $h(t) \in \Phi_C$, otherwise* unrestricted*.*

We shall think of the $\Im_1$ labels as possible worlds in a fibred (dovetailed) model, and labels in $\Im_n$ as paths leading to them starting from the actual world. For example a label such $t = (W_2^i, (w_2^j, w_0))$ denotes, with respect to a fibred (dovetailed) model, the set of worlds in a model for $L_i$ accessible from the actual world $a^i = \mathbf{F}_i((w_2^j, w_0))$. Since dovetailed models can be reduced to structures of the form $(W, R(i), h)$, the label $t$ turns out to represents the set of worlds accessible via $R(i)$ from a world accessible through $R(j)$ from the actual world $w_0$.

**Definition 7** *For any label t, we define the length of t, $\ell(t)$, as the number of world-symbols in t, i.e., $\ell(t) = n \Leftrightarrow t \in \Im_n$. $s^n(t)$ will denote the segment of t of length n, i.e., $s^n(t) = s(t)$ such that $\ell(s(t)) = n$. We shall use $h^n(t)$ as an abbreviations for $h(s^n(t))$. Notice that $h(t) = h^{\ell(t)}(t)$.*

**Definition 8** *For any label $t, \ell(t) > n$, we define the* counter-segment-$n$ *of t, as follows:*

$$c^n(t) = h(t) \times (\cdots \times (h^k(t) \times (\cdots \times (h^{n+1}(t), w_0)))) \ \ (n < k < \ell(t))$$

*where $w_0$ is a dummy label, i.e., a label not appearing in t (the context in which such a notion occurs will tell us what $w_0$ stands for. In most cases it will denote the actual world).*

The counter-segment-$n$ defines what remains of a given label after having identified the segment of length $n$ with a 'dummy' label $w_0$. The appropriate dummy label will be specified in the applications where such a notion is used. However, it can be viewed also as an independent atomic label. In the contest of fibring $w_0$ can be thought as denoting the actual world obtained via the fibring function from the world denoted by $s^n(t)$.

**Example 9** Given the label $t = (w_4, (W_3, (w_3, (W_2, w_1))))$, according to the above definitions its length $\ell(t)$ is 5, the head $h(t)$ is $w_4$, the body $b(t)$ is $(W_3, (w_3, (W_2, w_1)))$, the segment of length 3 is $s^3(t) = (w_3, (W_2, w_1))$, and the relative counter-segment-3 is $c^3(t) = (w_4, (W_3, w_0))$, where $w_0 = s^3(t) = (w_3, (W_2, w_1))$.

To clarify the notion of counter-segment, which will be used frequently in the course of the present work, we present, in the following table the list of the segments of $t$ in the left-hand column and the relative counter-segments in the right-hand column.

| | |
|---|---|
| $s^1(t) = w_1$ | $c^1(t) = (w_4, (W_3, (w_3, (W_2, w_0))))$ |
| $s^2(t) = (W_2, w_1)$ | $c^2(t) = (w_4, (W_3, (w_3, w_0)))$ |
| $s^3(t) = (w_3, (W_2, w_1))$ | $c^3(t) = (w_4, (W_3, w_0))$ |
| $s^4(t) = (W_3, (w_3, (W_2, w_1)))$ | $c^4(t) = (w_4, w_0)$ |
| $s^5(t) = t$ | $c^5(t) = w_0$ |

So far we have provided definitions about the structure of the labels without regard of the elements they are made of. The following definitions will be concerned with the type of world symbols occurring in a label.

Let $t$ be a label and $t'$ an atomic label, in what follows we shall use $(t'; t)$ as a notation for the label $(t', t)$ if $t' \neq h(t)$, or for $t$ otherwise

**Definition 10** *We say that a label $t$ is $i$*-preferred *iff $h(t) \in \Phi^i$.*

**Definition 11** *We say that a label $t$ is $i$*-pure *iff each segment of $t$ of length $n > 1$ is $i$-preferred, and we shall use $\Im^i$ to denote the set of $i$-pure labels.*

## 4.2 UNIFICATIONS

In the course of proofs labels are manipulated in a way closely related to the semantic of the logics under analysis. Labels are confronted and matched using a specialised logic dependent unification mechanism. The notion of two labels $t$ and $s$ unifying means that the intersection of their denotations is not empty and that we can move to such a set of worlds, i.e., to the result of their unification.

According to the semantics each modality is evaluated in an appropriate model corresponding to a model in the class of models characterising the logic the modality corresponds to. Similarly we provide an unification for each logic, the unification characterising such a logic in KEM formalism, then we graft them into a single unification for the whole $L_I^\delta$.

### 4.2.1 Basic Unifications (Axiom Unifications).

We add a set of auxiliary unindexed atomic labels $\Phi^A = \{w_0, w_0', \dots\}$, that will be used in unifications and proofs. Intuitively they stand for distinguished worlds in the various models. We define two substitutions, $\sigma^{\mathbf{D}}$ and $\sigma^{\mathbf{F}}$ resp. dovetailing and fibring substitution, in the usual way as a mapping

$$\begin{aligned} \sigma^\delta &= \mathbf{1}_{\Phi^A \cup \Phi_C} \\ \sigma^{\mathbf{D}} &: \Phi_V^i \longrightarrow \Im^i \cup \Phi^A \\ \sigma^{\mathbf{F}} &: \Phi_V^i \longrightarrow \Im^i \end{aligned}$$

i.e., identity for constants and auxiliary labels; and a mapping of variables onto $i$-pure labels in the case of fibring, and either onto $i$-pure labels or auxiliary ones for the other. The only difference between fibring and dovetailing substitutions is that, in the former, a variable cannot be mapped onto an auxiliary label. Henceforth we use $\sigma$ to mean indifferently, unless specified, either $\sigma^{\mathbf{D}}$ or $\sigma^{\mathbf{F}}$.

The substitution for composite labels is as follows: if $t = (s', s)$, then

$$\sigma(t) = (\sigma(s'), \sigma(s))$$

For two labels $t$ and $s$, and a substitution $\sigma$, if $\sigma$ is a unifier of $t$ and $s$ then we shall say that $t, s$ are *$\sigma$-unifiable*. We shall (somewhat unconventionally) use $[t|s]\sigma$ to denote both that $t$ and $s$ are $\sigma$-unifiable and the result of their unification. In particular

$$\forall t, s, r \in \Im, [t|s]\sigma = r \text{ iff } \exists\sigma(\sigma(t) = \sigma(s) \text{ and } \sigma(t) = r)$$

On this basis we may define several specialised, logic-dependent notions of $\sigma$-unification. As a case study we choose the normal modal logics arising from the combination of the axioms $K$, $D$, $T$, $4$, $B$, and $5$. Notice that the unifications listed below mimic the conditions on the accessibility relation corresponding to the appropriate axiom (see the accompanying examples for explanations).

$$[t|s]\sigma^K = [t|s]\sigma \quad \text{if at least one of } t \text{ and } s \text{ is restricted, and} \\ \forall n \leq \ell(t), [s^n(t)|s^n(s)]\sigma^K$$

$$[t|s]\sigma^D = [t|s]\sigma$$

**Example 12** To exemplify the difference between $\sigma^K$ and $\sigma^D$, let us consider first the labels

$$(w_3, (W_1, w_1)) \qquad\qquad (W_2, (w_2, w_1))$$

Obviously $t$ and $s$ $\sigma^K$- and $\sigma^D$-unify on $(w_3, (w_2, w_1))$ with the substitution

$$\sigma : W_1 \mapsto w_2 \\ W_2 \mapsto w_3$$

On the other hand the labels

$$t = (w_2, (W_1, w_1)) \qquad\qquad s = (W_2, (W_1, w_1))$$

$\sigma^D$- but not $\sigma^K$-unify. This is due to the fact that both $s^2(t)$ and $s^2(s)$ are variables, while in the definition of $\sigma^K$ it is required that at least one of them is a constant. The reason for this condition on $\sigma^K$ is that the interpretation of $W_1$ is the set of worlds accessible from $w_1$, but such a set may be empty so the denotation of $W_1$ would be empty; this is not the case with $\sigma^D$ since the corresponding accessibility relation is serial, so $W_1$ cannot be empty.

$$[t|s]\sigma^T = \begin{cases} [s^{\ell(s)}(t)|s]\sigma & \text{if } \ell(t) > \ell(s), \text{ and} \\ & \forall n \geq \ell(s), [h^n(t)|h(s))]\sigma = [h(t)|h(s)]\sigma \\ [t|s^{\ell(t)}(s)]\sigma & \text{if } \ell(s) > \ell(t), \text{ and} \\ & \forall n \geq \ell(t), [h(t)|h^n(s)]\sigma = [h(t)|h(s)]\sigma \end{cases}$$

**Example 13** For the notion of $\sigma^T$-unification, take for example the labels

$$t = (w_3, (W_1, w_1)) \qquad\qquad s = (w_3, (W_2, (w_2, w_1)))$$

Here $[W_2|w_3]\sigma = [w_3|w_3]\sigma$. Then the two labels $\sigma^T$-unify to $(w_3, (w_2, w_1))$. This intuitively means that the world $w_3$, accessible from a sub-path $s(s) =$

$(W_2, (w_2, w_1))$, after the deletion of $W_2$ from $s$, is accessible from any path $t$ which turns out to denote the same world(s) as $s(s)$; in fact the step from $w_2$ to $W_2$ is irrelevant because of the reflexivity relation of the model.

$$[t|s]\sigma^4 = \begin{cases} c^{\ell(t)}(s) & \ell(s) > \ell(t), h(t) \in \Phi_V \text{ and} \\ & w_0 = [t|s^{\ell(t)}(s)]\sigma \\ c^{\ell(s)}(t) & \ell(t) > \ell(s), h(s) \in \Phi_V \text{ and} \\ & w_0 = [s^{\ell(s)}(t)|s]\sigma \end{cases}$$

**Example 14** For the notion of $\sigma^4$-unification, take for example the labels

$$t = (W_3, (w_2, w_1)) \qquad s = (w_5, (w_4, (w_3, (W_2, w_1))))$$

Here $s^{\ell(t)}(s) = (w_3, (W_2, w_1))$. Then $t$ and $s$ $\sigma^4$-unify to

$$(w_5, (w_4, (w_3, (w_2, w_1))))$$

since

$$[t|s^{\ell(t)}(s)]\sigma = [(W_3, (w_2, w_1))|(w_3, (W_2, w_1))]\sigma .$$

This intuitively means that all the worlds accessible from a sub-path $s^{\ell(t)}(s)$ of $s$ are accessible from any path $t$ which leads to the same world(s) denoted by $s^{\ell(t)}(s)$. Here $W_3$ stands for the set of worlds accessible from $w_2$; Then $w_3$, after the unification of $(w_2, w_1)$ and $(W_2, w_1)$, is one of such worlds. $w_4$ is accessible from $w_3$ and, via transitivity, from $w_2$. The same for $w_5$.

$$[t|s]\sigma^B = \begin{cases} [s^{\ell(t)-2n}(t)|s]\sigma & \text{if } h(t) \in \Phi_V \text{ and} \\ & [h(t)|h(s)]\sigma = [h^{\ell(t)-2n}(t)|h(s)]\sigma \\ [t|s^{\ell(s)-2n}(s)]\sigma & \text{if } h(s) \in \Phi_V \text{ and} \\ & [h(t)|h(s)]\sigma = [h(t)|h^{\ell(s)-2n}(s)]\sigma \end{cases}$$

Where $1 \leq n \leq V$, and $V = \ell(t) - m$, with $m$ such that $\forall x, m \leq x \leq \ell(t), h^x(t) \in \Phi_V$.

**Example 15** For $\sigma^B$ we consider the labels

$$t = (W_3, (W_2, (w_2, (W_1, w_1)))) \qquad s = (W_4, (w_3, w_1)) \tag{7.1}$$

The labels $t$ and $s$ $\sigma^B$-unify since $t$ has two variables, so we have two chances of going back: one steps from $b(t)$, or two steps from $b(b(t))$. In the first case we have to see whether $(w_2, (W_1, w_1)) = s^{\ell(t)-2n}(t), n = 1$ and $s$ $\sigma$-unify. In the second case the label that have to $\sigma$-unify with $s$ is $w_1 = s^{\ell(t)-2n}(t), n = 2$. But in this case the unification fails. The key idea of $\sigma^B$-unification is to match world symbols laying an even number of steps apart. The number of steps is given by the number of consecutive variables present in the labels. If the head of a label is a variable we can go back by two steps. In general we are allowed to return back of two steps for each variable. Labels like $(W_1, (w_2, w_1))$ and $w_1$ are a simple instance of such an unification. $W_1$ denotes the set of worlds accessible from $w_2$, but, since $w_2$ is accessible from $w_1$; so, by symmetry, $w_1$ is one of the world accessible from $w_2$.

$$[t|s]\sigma^5 = \begin{cases} ([h(t)|h(s)]\sigma; c^1(s^2(t))) & \text{if } \ell(t) > 2, \ell(s) > 1, h(t) \in \Phi_V, \text{ or} \\ & h(t) = h(s) \in \Phi_C \\ [t|s]\sigma & \text{if } \ell(t) = \ell(s) = 2 \\ ([t|h(s)]\sigma; c^1(s^2(s))) & \text{if } \ell(s) > 2, \ell(t) > 1, h(s) \in \Phi_V, \text{ or} \\ & h(t) = h(s) \in \Phi_C \end{cases}$$

where $w_0 = [s^1(t)|s^1(s)]\sigma$.

We exemplify how unifications corresponding to axioms obtained from the axioms listed above by prefixing $\Box^n$, $n \in \mathbb{N}$, to them can be defined.

$$[t|s]\sigma^O = [c^2(t)|c^2(s)]\sigma^D$$

where $w_0 = [s^2(t)|s^2(s)]\sigma^K$, and $O = \Box(\Box A \to \Diamond A)$.

**4.2.2 High unifications (combined unifications).** We are now able to combine the above unifications corresponding to the axiom characterising a logic into a single 'high' unification which will be used for defining the unifications characterising the logic we are concerned with.

$$[t|s]\sigma^{A_1^i \cdots A_n^i} = \begin{cases} [t|s]\sigma^{A_1^i} & \text{if } C_1^i \\ \vdots & \vdots \\ [t|s]\sigma^{A_n^i} & \text{if } C_n^i \end{cases} \qquad i \in I \tag{7.2}$$

where $A_1^i \cdots A_n^i$ stand for the axioms characterising $L_i$ and $C_i, 1 \leq i \leq n$ are conditions varying from logic to logic. For example the high unification for **T**, which is characterized by the axioms $D$ and $T$, is

$$[t|s]\sigma^{DT} = \begin{cases} [t|s]\sigma^T & \text{if } \ell(t) \neq \ell(s) \\ [t|s]\sigma^D & \text{otherwise} \end{cases}$$

and for **OM** which is **K** plus $M = \Box(\Box A \to A)$, the deontic version of **T**, the corresponding high unifications is

$$[t|s]\sigma^{OM} = \begin{cases} [t|s]\sigma^T & \text{if } [s^2(t)|s^2(s)]\sigma^O \\ [t|s]\sigma^O & \end{cases}$$

We then provide the definition of $\sigma^{DT4}$ which is used in defining the logic unification for **S4**.

$$[t|s]\sigma^{DT4} = \begin{cases} [t|s]\sigma^D & \text{if } \ell(t) = \ell(s) \\ [t|s]\sigma^T & \text{if } \ell(t) < \ell(s), h(t) \in \Phi_C \\ [t|s]\sigma^4 & \text{if } \ell(t) < \ell(s), h(t) \in \Phi_V \end{cases} \tag{7.3}$$

It is worth noting that the conditions on axiom unifications are needed in order to provide a deterministic unification procedure. In general, if axiom unifications are given for each axiom characterising a logic, such conditions are not necessary.

**4.2.3 Low unification (logic unifications).** Combining recursively each high unification we obtain the $\sigma_{L_i}$ unification for $L_i$ as follows

$$[t|s]\sigma_{L_i} = \begin{cases} [c^n(t)|c^m(s)]\sigma^{A_1^i \cdots A_n^i} \\ [t|s]\sigma^{A_1^i \cdots A_n^i} \end{cases} \tag{7.4}$$

where $w_0 = [s^n(t)|s^m(s)]\sigma_{L_i}$.

Although the above definition provides a general method for obtaining the unification characterising the appropriate logic, for particular systems it may be preferred to define a different unification reflecting peculiar properties; for example, for **S5** which is characterised by the class of frames where the accessibility relation is an equivalence relation, it is convenient to define $\sigma_{\mathbf{S5}}$ as follows:[2]

$$[t|s]\sigma_{\mathbf{S5}} = \begin{cases} [h(t)|h(s)]\sigma & \min\{\ell(t), \ell(s)\} = 1 \\ ([h(t)|h(s)]\sigma, [s^1(t)|s^1(s)]\sigma) & \text{otherwise} \end{cases}$$

The format of the unifications for the logics containing axioms obtained by prefixing $\Box^n$ to one of the five axiom listed above is slightly more complex. We exemplify it by providing the unification for **OM**.

$$[t|s]\sigma_{\mathbf{OM}} = \begin{cases} [c^n(t)|c^m(s)]\sigma^{OM} \\ [c^n(t)|c^m(s)]\sigma^{T} & n, m > 2 \\ [t|s]\sigma^{OM} \end{cases}$$

where $w_0 = [s^n(t)|s^m(s)]\sigma_{\mathbf{OM}}$.

**Example 16** According to (7.4) the logic unification for **S4** is

$$[t|s]\sigma_{\mathbf{S4}} = \begin{cases} [c^n(t)|c^m(s)]\sigma^{DT4} \\ [t|s]\sigma^{DT4} \end{cases}$$

where $w_0 = [s^n(t)|s^m(s)]\sigma_{\mathbf{S4}}$.

We provide a pair of labels $t$, and $s$ that $\sigma_{\mathbf{S4}}$-unify.

$$t = (w_6, (w_5, (W_2, (w_1, (W_1, w_0))))) \quad s = (W_3, (w_4, (w_3, (w_2, (w_1, w_0)))))$$

---

[2]Similar considerations hold for **K45** and **D45**. So specialised unifications for such logics have been provided in [16]. However in [18] equivalence between these specialised unifications and unifications obtained from the schema 7.4 has been proved.

The two labels do not $\sigma^D$-, nor $\sigma^T$-, nor $\sigma^4$-unify. We have then split them into appropriate counter-segments.

Graphically we can represent the steps of above unification as follows:

$$\begin{array}{r} s = \\ t = \end{array} \underbrace{\underbrace{\begin{array}{l} W_3, \\ w_6, w_5, \end{array}}_{\sigma^4} w_0'' = \left[ \underbrace{\underbrace{\begin{array}{c} w_4, w_3, w_2, \\ W_2 \end{array}}_{\sigma^4} w_0' = \left[ \underbrace{\begin{array}{r} w_1, w_0 \\ w_1, W_1, w_0 \end{array}}_{\sigma^T} \right]}_{\sigma_{\mathbf{S4}}} \right]}_{\sigma_{\mathbf{S4}}}$$

In the first line we show the elements of the label $s$; the second line contains the elements of $t$. Notice that here $w_0'$ and $w_0''$ are dummy labels and are common to $t$ and $s$.

First we consider

$$s^3(t) = (w_1, (W_1, w_0)) \qquad \text{and} \qquad s^2(s) = (w_1, w_0)\ .$$

Such labels $\sigma^T$-unify on $(w_1, w_0)$, see example 13. We identify their unification with $w_0'$; then

$$c^3(t) = (w_6, (w_5, (W_2, w_0'))) \quad \text{and} \quad c^2(s) = (W_3, (w_4, (w_3, (w_2, w_0'))))\ .$$

At this point we notice that

$$s^2(c^3(t)) = (W_2, w_0') \qquad s^4(c^2(s)) = (w_4, (w_3, (w_2, (w_1, w_0))))$$

$\sigma^4$-unify: $[w_0'|w_0']\sigma$ and $W_2 \in \Phi_V$. The result of their unification, $w_0''$, is $(w_4, (w_3, (w_2, (w_1, w_0))))$. Finally

$$c^4(t) = (w_6, (w_5, w_0'')) \qquad c^5(s) = (W_3, w_0'')$$

$\sigma^4$-unify. According to the definition of logic unification 7.4, and the previous considerations $t$ and $s$ $\sigma_{\mathbf{S4}}$-unify.

**Example 17** In (7.1) we have seen a pair of labels that $\sigma^B$. Here we provide two labels that unify by applying recursively $\sigma^B$.

$$i = (W_3, (w_4, (W_2, (w_3, (W_1, (w_2, w_1)))))) \qquad k = w_1 \tag{7.5}$$

The unification is obtained according to the following decomposition.

$$[(W_3, (w_4, w_0))|w_1]\sigma^{KB}$$

where

$$w_0 = [(W_2, (w_2, w_0'))|w_1]\sigma_{\mathbf{KB}}$$

and

$$w_0' = [(W_1, (w_2, w_1))|w_1]\sigma_{\mathbf{KB}}$$

since

$$[(W_1, (w_2, w_1))|w_1]\sigma^{KB} = w_1$$

by an immediate application of $\sigma^B$.

For a wide class of unifications and a detailed account of the labelling algebra of KEM see [18].

**4.2.4 Fibred unification.** As $L_I^\delta$ is obtained by combining the logic $L_i, i \in I$ the corresponding unification is the combination of the $\sigma_{L_i}$-unifications characterising $L_i$.

$$[t|s]\sigma_{L_I^\delta} = \begin{cases} \bigcup_{i\in I}[c^n(t)|c^m(s)]\sigma_{L_i} \\ \bigcup_{i\in I}[t|s]\sigma_{L_i} \end{cases} \tag{7.6}$$

where $w_0 = [s^n(t)|s^m(s)]\sigma_{L_I^\delta}$, if $c^n(t), c^n(s)$ are $i$-pure, $i \in I$.

It is worth noting that the mechanism of the fibred unification is, essentially, the same mechanism governing the logic unification.

**Example 18** We want to combine a deontic **D** modality $\Box_1$ with an epistemic **S5** modality $\Box_2$, let us call the resulting logic $\mathbf{ED}^\delta$. The unifications for $\mathbf{ED}^\delta$ are provided by the schema

$$[t|s]\sigma_{\mathbf{E}D^\delta} = \begin{cases} [c^n(t)|c^m(s)]\sigma_{\mathbf{S5^2}} \\ [t|s]\sigma_{\mathbf{S5^2}} \\ [c^n(t)|c^m(s)]\sigma_{\mathbf{D^1}} \\ [t|s]\sigma_{\mathbf{D^1}} \end{cases}$$

where $w_0 = [s^n(t)|s^m(s)]\sigma_{\mathbf{ED}^\delta}$, if $c^n(t)$ and $c^n(s)$ are $i$-pure, $i \in \{1, 2\}$.

According to the above definition, the labels

$$t = (W_2^2, (w_1^2, (W_1^1, w_0))) \qquad s = (w_2^2, (W_2^1, w_0))$$

$\sigma_{\mathbf{ED}^F}$-unify, and then $\sigma_{\mathbf{ED}^D}$-unify. In fact

$$c^2(t) = (W_2^2, (w_1^2, w_0') \qquad c^2(s) = (w_2^2, w_0')$$

$\sigma_{\mathbf{S5^2}}$-unify, and $w_0' = [(W_1^1, w_0)|(w_1^1, w_0)]\sigma_{\mathbf{D^1}}$. On the contrary, if $t$ had been $t' = (W_2^2, (w_2^1, (w_1^2, (W_1^1, w_0))))$, it would not have unified with $s$: $c^2(t')$ is not 2-pure. Let us consider now the labels

$$t = (W_1^1, w_0) \qquad s = (W_1^2, (w_1^2, (W_2^1, w_0)))$$

$t$ and $s$ do not $\sigma_{\mathbf{ED}^F}$-unify, but $\sigma_{\mathbf{ED}^D}$-unify, in so far as we can map $W_2^2$ onto $w'_0$, where $w'_0$ is as before.

We prove now an interesting property of $\sigma_{L_I^\delta}$-unifications.

**Lemma 19** $\forall t, s \in \Im$, *if* $[t|s]\sigma_{L_I^\delta} = r$, *then* $[t|r]\sigma_{L_I^\delta}$ *and* $[r|s]\sigma_{L_I^\delta}$.

**Proof** Since $\sigma_{L_I^\delta}$ has been defined as the combination of $\sigma_{L_i}$-unifications, but each of them works on $i$-pure labels and there are no interactions among them, we just have to prove the property for each $\sigma_{L_i}$.

The proof will be by induction on the number of applications of $\sigma^{A_1^i \cdots A_n^i}$ in a $\sigma_{L_i}$-unification. Let $n$ be the number of such applications. We give the proof only for **S4** and **S5**. The proof for the others systems is similar and can be found in [1, 18].

If $n = 1$ then we have to prove the property for $\sigma^{A_1^i \cdots A_n^i}$.[3] We prove it by induction on the length of labels.

If $\min\{\ell(t), \ell(s)\} = 1$ then we assume that $\ell(t) = 1$ (the proof for $\ell(s) = 1$ is similar). 1) $t \in \Phi_C$. If also $\ell(s) = 1$, we apply $\sigma^D$; in every case, by obvious considerations about $\sigma$, $r = [t|s]\sigma^D = t$, but $[t|t]\sigma^D$ and $[t|s]\sigma^D$. Therefore $[t|r]\sigma^{DT4}$ and $[r|s]\sigma^{DT4}$. If $\ell(s) > 1$ then $[t|s]\sigma^{DT4} = [t|s]\sigma^T$. Therefore $r = [t|s]\sigma^T = [t|s^1(s)]\sigma_T = t$, hence $[t|t]\sigma^D$ and $[t|s]\sigma^T$, and so, according to 7.3, $[t|r]\sigma^{DT4}$ and $[r|s]\sigma^{DT4}$.

2) $t \in \Phi_V$ then by the definition of $\sigma$ it unifies with any label, in particular $[t|s]\sigma^D = s = r$, whence $[t|s]\sigma^D$ and $[s|s]\sigma^D$, then $[t|r]\sigma^{DT4}$ and $[r|s]\sigma^{DT4}$.

Let us suppose now that $\min\{\ell(t), \ell(s)\} = n > 1$, and that the property holds up to $n$ for $\sigma^{A_1^i \cdots A_n^i}$. Thus we have the following cases.

If $\ell(t) = \ell(s)$ then $[t|s]\sigma^D = s$; by the inductive hypothesis $[b(t)|b(l)]\sigma^D$, $[b(s)|b(r)]\sigma^D$, $[h(t)|h(s)]\sigma^D$ and $[h(s)|h(r)]\sigma^D$; hence $[t|r]\sigma^D$ and $[s|r]\sigma^D$. Consequently $[t|r]\sigma^{DT4}$ and $(r, s)^{DT4}$.

If $\ell(t) < \ell(s)$ and $h(t) \in \Phi_C$, then $[t|s]\sigma^T = r$, where, by the inductive hypothesis $[b(t)|b(l)]\sigma^D$, $[s^{\ell(b(t))}(s)|b(r)]\sigma^D$. By the definition of $\sigma^T$, we know that, for $n \leq \ell(t)$ $h^n(r) = [h(t)|h(s)]\sigma = [h(t)|h^{\ell(t)}(s)]\sigma$; therefore $[t|r]\sigma^D$ and $[r|s]\sigma^T$. We can conclude that $[t|r]\sigma^{DT4}$ and $[r|s]\sigma_{DT4}$.

If $\ell(t) < \ell(s)$ and $h(t) \in \Phi_V$, then $[t|s]\sigma^4 = c^{\ell(t)}(s)$ where $w'_0 = [t|s^{\ell(t)}(s)]\sigma$. By the inductive hypothesis and the definition of $\sigma$ we have $[t|s^{\ell(t)}(r)]\sigma$ and $[s^{\ell(t)}(t)|s^{\ell(t)}(r)]\sigma$ and therefore $[t|r]\sigma^4$ and $[r|s]\sigma^D$, which means $[t|r]\sigma^{DT4}$ and $[r|s]\sigma^{DT4}$.

We have thus proved the inductive base for the lemma.

[3] Hereafter, in order to shorten proofs, when we have to consider labels of different lengths, we shall assume, unless specified, the first to be the shorter. Obviously proofs for the other cases can be carried out in the same way.

We can now assume that the lemma holds up to the $n$-th application of $\sigma^{A_1^i \cdots A_n^i}$. By the definition of $\sigma_{L_i}$, $[s^n(t)|s^m(s)]\sigma_{L_i} = w_0' = s^r(r)$ and $[c^n(t)|c^m(s)]\sigma^{A_1^i \cdots A_n^i} = c^r(r)$; but, by the inductive hypothesis, we know $[s^n(t)|s^r(r)]\sigma_{L_i}$ and $[s^m(s)|s^r(r)]\sigma_{L_i}$. By the property we have just proved for $\sigma^{A_1^i \cdots A_n^i}$ we obtain $[c^n(t)|c^r(r)]\sigma^{A_1^i \cdots A_n^i}$ and $[c^m(s)|c^r(r)]\sigma^{A_1^i \cdots A_n^i}$, which implies $[t|r]\sigma_{L_i}$ and $[r|s]\sigma_{L_i}$.

For **S5** if $\min\{\ell(i), \ell(s)\} = 1$, we have $[t|s]\sigma_{\mathbf{S5}}$ iff $[h(t)|h(s)]\sigma$, whence, if $t$ is restricted, then $[t|s]\sigma_{\mathbf{S5}} = h(t) = r$ and thus $[t|s]\sigma_{\mathbf{S5}}$, i.e., $[h(t)|h(t)]\sigma$, and similarly for $s$; otherwise $[t|s]\sigma_{\mathbf{S5}} = h(s) = r$, therefore for the same reason as in the previous case $[r|s]\sigma_{\mathbf{S5}}$ and $[t|r]\sigma_{\mathbf{S5}}$. If $\min\{\ell(t), \ell(s)\} > 1$ we can repeat the same argument of the other case with the difference that $r = ([h(t)|h(s)]\sigma, [s^1(t)|s^1(s)]\sigma)$. ■

**Remark 20** *It is worth noting that we have no constraints on the component logics; they may be combined logics themselves.*

## 4.3 INFERENCE RULES

In displaying the rules of KEM we shall use Smullyan-Fitting [10] $\alpha$, $\beta$, $\nu_i$, $\pi_i$, $i \in I$ unifying notation as exposed in the following tables:

| $\alpha$ | $\alpha_1$ | $\alpha_2$ |
|---|---|---|
| $\mathsf{T}A \wedge B$ | $\mathsf{T}A$ | $\mathsf{T}B$ |
| $\mathsf{F}A \vee B$ | $\mathsf{F}A$ | $\mathsf{F}B$ |
| $\mathsf{F}A \to B$ | $\mathsf{T}A$ | $\mathsf{F}B$ |
| $\mathsf{F}\neg A$ | $\mathsf{T}A$ | $\mathsf{T}A$ |

| $\beta$ | $\beta_1$ | $\beta_2$ |
|---|---|---|
| $\mathsf{F}A \wedge B$ | $\mathsf{F}A$ | $\mathsf{F}B$ |
| $\mathsf{T}A \vee B$ | $\mathsf{T}A$ | $\mathsf{T}B$ |
| $\mathsf{T}A \to B$ | $\mathsf{F}A$ | $\mathsf{T}B$ |
| $\mathsf{T}\neg A$ | $\mathsf{F}A$ | $\mathsf{F}A$ |

The relationships between $\alpha$- and $\beta$-formula are

$$\begin{array}{lll} \alpha = \beta^C & \alpha_1 = \beta_1^C & \alpha_2 = \beta_2^C \\ \beta = \alpha^C & \beta_1 = \alpha_1^C & \beta_2 = \alpha_2^C. \end{array}$$

Formulas of type $\pi_i$ and $\nu_i$ are classified as follows

| $\nu_i$ | $\nu_0$ |
|---|---|
| $\mathsf{T}\Box_i A$ | $\mathsf{T}A$ |
| $\mathsf{F}\Diamond_i A$ | $\mathsf{F}A$ |

| $\pi_i$ | $\pi_0$ |
|---|---|
| $\mathsf{F}\Box_i A$ | $\mathsf{F}A$ |
| $\mathsf{T}\Diamond_i A$ | $\mathsf{T}A$ |

Similarly we provide the relationships between $\nu_i$- and $\pi_i$-formulas.

$$\begin{array}{ll} \nu_i = \pi_i^C & \nu_0 = \pi_0^C \\ \pi_i = \nu_i^C & \pi_0 = \nu_0^C. \end{array}$$

Given a signed formula $X$, $X^C$ denotes the *conjugate* of $X$, i.e., the result of changing the sign of $X$ to its opposite; two $LS$-formulas $X : t$ and $X^C : s$ such that $[t|s]\sigma_{L_I^\delta}$ and will be called $\sigma_{L_I^\delta}$-$complementary$.

$$\frac{\alpha : t}{\alpha_1 : t} \qquad \frac{\alpha : t}{\alpha_2 : t} \tag{$\alpha$}$$

$$\frac{\begin{array}{c}\beta : t \\ \beta_1^C : s\end{array}}{\beta_2 : [t|s]\sigma_{L_I^\delta}}\ [t|s]\sigma_{L_I^\delta} \qquad \frac{\begin{array}{c}\beta : t \\ \beta_2^C : s\end{array}}{\beta_1 : [t|s]\sigma_{L_I^\delta}}\ [t|s]\sigma_{L_I^\delta} \tag{$\beta$}$$

$$\frac{\nu_i : t}{\nu_0 : (s,t)}\ s \in \Phi_V^i \text{ and new} \tag{$\nu_i$}$$

$$\frac{\pi_i : t}{\pi_0 : (s,t)}\ s \in \Phi_C^i \text{ and new} \tag{$\pi_i$}$$

$$\frac{}{X : t \qquad\qquad X^C : t}\ t \text{ restricted} \tag{PB}$$

$$\frac{\begin{array}{c}X : t \\ X^C : s\end{array}}{\times [t|s]\sigma_{L_I^\delta}}\ [t|s]\sigma_{L_I^\delta} \tag{PNC}$$

Here the $\alpha$-rules are just the familiar linear branch-expansion rules of the tableau method, while the $\beta$-rules correspond to such common natural inference patterns as *modus ponens*, *modus tollens*, etc. The rules for the modal operators are as usual. '$s$ new' in the proviso for the $\nu_i$- and $\pi_i$-rule means: $s$ must not have occurred in any label yet used. Notice that in all inferences via an $\alpha$-rule the label of the premise carries over unchanged to the conclusion, and in all inferences via a $\beta$-rule the labels of the premises must be $\sigma_{L_I^\delta}$-unifiable, so that the conclusion inherits their unification. *PB* (the 'Principle of Bivalence') represents the (*LS*-version of the) semantic counterpart of the cut rule of the sequent calculus (intuitive meaning: a formula $A$ is either true or false in any *given* world, whence the requirement that $t$ should be restricted). *PNC* (the 'Principle of Non-Contradiction') corresponds to the familiar branch-closure rule of the tableau method, saying that from the occurrence of a pair of $\sigma_{L_I^\delta}$-complementary formulas on a branch we may infer the closure ('$\times$') of the branch. The $[t|s]\sigma_{L_I^\delta}$ in the 'conclusion' of *PNC* means that the contradiction holds 'in the same world'. Other logics might require additional rules in order to capture the full power of their semantics. See for example [17, 18]. As usual with refutation methods a KEM-proof of $A$ consists of a successful attempt to construct a counter model for $A$ by assuming that $A$ is false in some arbitrary model, which means that we assume that $A$ is false in the actual world of the model. So a KEM-proof in $L$ for $A$ ($\vdash_{\mathrm{KEM}(L)} A$)is a closed tree starting with $\mathsf{F}A$. A tree is closed iff all its branch are closed; a branch is closed iff it contains an application of *PNC*.

Let us consider the combined systems $\mathbf{ED}^{\delta}$ of example 18 and the formulas $A = \Box_1\Diamond_2 p \to \Diamond_1\Box_2\Diamond_2 p$ and $B = \Box_1 p \to \Diamond_1\Box_2\Diamond_2 p$. It is easy to see that $A$ is valid in $\mathbf{ED}^{\delta}$. On the other hand, $B$ holds in $\mathbf{ED}^{D}$, but fails in $\mathbf{ED}^{F}$. We provide now their KEM-proofs, and, from them, the fibred model where $B$ fails can be easily obtained.

| | | |
|---|---|---|
| 1. | $\mathsf{F}\Box_1\Diamond_2 p \to \Diamond_1\Box_2\Diamond_2 p$ | $w_0$ |
| 2. | $\mathsf{T}\Box_1\Diamond_2 p$ | $w_0$ |
| 3. | $\mathsf{F}\Diamond_1\Box_2\Diamond_2 p$ | $w_0$ |
| 4. | $\mathsf{T}\Diamond_2 p$ | $(W_1^1, w_0)$ |
| 5. | $\mathsf{F}\Box_2\Diamond_2 p$ | $(W_2^1, w_0)$ |
| 6. | $\mathsf{T}p$ | $(w_1^2, (W_1^1, w_0))$ |
| 7. | $\mathsf{F}\Diamond_2 p$ | $(w_2^2, (W_2^1, w_0))$ |
| 8. | $\mathsf{F}p$ | $(W_1^2, (w_2^2, (W_2^1, w_0)))$ |

| | | |
|---|---|---|
| 1. | $\mathsf{F}\Box_1 p \to \Diamond_1\Box_2\Diamond_2 p$ | $w_0$ |
| 2. | $\mathsf{T}\Box_1$ | $w_0$ |
| 3. | $\mathsf{F}\Diamond_1\Box_2\Diamond_2 p$ | $w_0$ |
| 4. | $\mathsf{T}p$ | $(W_1^1, w_0)$ |
| 5. | $\mathsf{F}\Box_2\Diamond_2 p$ | $(W_2^1, w_0)$ |
| 6. | $\mathsf{F}\Diamond_2 p$ | $(w_2^2, (W_2^1, w_0))$ |
| 7. | $\mathsf{F}p$ | $(W_1^2, (w_2^2, (W_2^1, w_0)))$ |

Steps 2, and 3 are obtained by an application of an $\alpha$-rule on 1, the other steps have been obtained by straightforward applications of $\nu_i$- and $\pi_i$-rules, until we reach atomic formulas. We notice that in both trees we have complementary formulas, i.e., 6, 8 in the tree for $A$ and 4, 7 in the tree for $B$. All that remains to do is verifying whether they are $\sigma_{\mathbf{ED}^{\delta}}$-complementary; i.e., we have to check if their labels $\sigma_{\mathbf{ED}^{\delta}}$-unify. In example 18 we have seen that the labels of 6 and 8 $\sigma_{\mathbf{ED}^{\delta}}$-unify. However these of 4 and 7 $\sigma_{\mathbf{ED}^{D}}$- but not $\sigma_{\mathbf{ED}^{F}}$-unify.

## 4.4 SOUNDNESS AND COMPLETENESS

In this section we prove soundness and completeness for KEM-based fibred modal tableaux. There are very similarity with the results of [5], however the material presented in this section points out close connections between unifications and modal fibring.

**Theorem 21** *If a* KEM *tree closes it closes atomically.*

**Proof** A closed KEM tree means that each branch is closed, i.e., it contains two $\sigma_{L_I^\delta}$-complementary formulas $A : t$ and $A^C : s$.

$$\begin{array}{c} \vdots \\ A : t \\ \vdots \\ A^C : s \\ \times \end{array}$$

We prove the theorem by induction of the complexity of the complementary formulas. If they are literals then the branch closes atomically.

If they are not literals let us examine their form: if $A$ is of type $\alpha$ then $A^C$ is of type $\beta$; moreover $\alpha_1 = \beta_1^C$ and $\alpha_2 = \beta_2^C$. We apply an $\alpha$-rule on $A : t$, obtaining $\alpha_1 : t$ and $\alpha_2 : t$. Since the relations just mentioned we can apply a $\beta$-rule w.r.t. $A^C : s$ and $\alpha_n$ $(n = 1, 2)$, from which we derive $\beta_{3-n} : [t|s]\sigma_{L_I^\delta}$. At this point the branch contains $\alpha_n : t$ and $\beta_n : [t|s]\sigma_{L_I^\delta}$, which are $\sigma_{L_I^\delta}$-complementary, in so far as $[t|[t|s]\sigma_{L_I^\delta}]\sigma_{L_I^\delta}$, see Lemma 19. If $A$ is of type $\beta$ we repeat the above reasoning applying the $\alpha$-rule on $A^C$ instead of $A$.

If $A$ is of type $\nu_i$, then $A^C$ is of type $\pi_i$ and $\nu_0 = \pi_0^C$. We apply a $\nu_i$-rule on $A : t$ and a $\pi_i$-rule on $A^C : s$ obtaining $\nu_0 : (W_n, t)$ and $\pi_0 : (w_m, s)$, where $W_n$ and $w_m$ are new in the branch. The resulting formulas are $\sigma_{L_I^\delta}$-complementary due to the relationship between $\nu_i$ and $\pi_i$ formulas and the fact that the labels obviously $\sigma_{L_I^\delta}$-unify. If $A$ is of type $\pi_i$, then $A^C$ is of type $\nu_i$ and we can repeat the same argument. ■

In the course of KEM-proofs labels are used to build appropriate models. Since the structure of the labels and unifications follows closely that of dovetailed and fibred models, we can repeat the same construction 'grafting' the models for each $L_i$ through $\mathbf{F}$ into fibred and dovetailed models obtaining models for $L_I^\delta$.

**Theorem 22** *Let $L_i, i \in I$ be modal logics and let $L_I^\delta$ the resulting combined logic. If* $\models_{\mathcal{K}_i} A \iff \vdash_{\mathrm{KEM}(L_i)} A$ *then*

1. $\models_{L_I^F} A \iff \vdash_{\mathrm{KEM}(L_I)} A$ *using* $\sigma^{\mathbf{F}}$*;*

2. $\models_{L_I^D} A \iff \vdash_{\mathrm{KEM}(L_I)} A$ *using* $\sigma^{\mathbf{D}}$.

**Proof** We prove the theorem by showing a) the set $\mathbf{T}_I^\delta$ is a subset of set of formulas provable in KEM and b) KEM rules are sound with respect to fibred and dovetailed models.

We start proving a). According to Theorems 3 and 4 a formula is a fibred (dovetailed) theorem if either it is a theorem of a component, or has been obtained by one of the axioms, or has been derived from an application of modal fibring (dovetailing) rule or modus ponens. We have then to prove that such rules and axioms are derived in KEM.

By hypothesis $\mathbf{T}_i$ coincides with the set of formulas provable in KEM for $L_i$.

For axiom 1b1 (Theorem 3), by hypothesis $A \rightarrow \alpha_n$ is a theorem of a fibred language, therefore $\mathsf{T}A : w_0$ and $\mathsf{F}\alpha_n : w_0$ lead to a closed KEM-tree. Let us start now a KEM-tree for $A \rightarrow \bigvee_n \alpha_n : w_0$, we obtain

$$\begin{array}{c} \mathsf{F}A \rightarrow \bigvee_n \alpha_n : w_0 \\ \mathsf{T}A : w_0 \\ \mathsf{F}\bigvee_n \alpha_n : w_0 \\ \mathsf{F}\alpha_1 : w_0 \\ \vdots \\ \mathsf{F}\alpha_n : w_0 \end{array}$$

At this point we can graft the proof for $A \rightarrow \alpha_n$, closing thus the tree.

For axiom 1b2 (Theorem 3), by hypothesis $A(x_j)$ has a closed KEM-tree, which means that each branch $\tau$ is closed; Theorem 21 implies that each branch is atomically closed, therefore each branch contains two $\sigma_{L_i}$-complementary labelled signed literals, let us say $x_\tau : t_\tau$ and $x_\tau^C : s_\tau$. We can now replace $\Box_j \alpha_j$ to $x_\tau$ obtaining $S\Box_j \alpha_j : t_\tau$ and $S^C \Box_j \alpha_j : s_\tau$. But the last two formulas are $\sigma_{L_I^F}$-complementary, then also in this case the tree is closed.

For Modus Ponens. By hypothesis $A$ and $A \rightarrow B$ have closed KEM trees, let us call them $\mathcal{T}_1$ and $\mathcal{T}_2$.

$$\begin{array}{ccc} & \mathsf{T}B : w_0 & \\ \swarrow & & \searrow \\ \begin{array}{c} \mathsf{T}A \rightarrow B : w_0 \\ \mathsf{F}A : w_0 \\ \mathcal{T}_1 : w_0 \\ \times \end{array} & & \begin{array}{c} \mathsf{F}A \rightarrow B : w_0 \\ \mathcal{T}_2 : w_0 \\ \times \\ \phantom{x} \end{array} \end{array}$$

For modal and dovetailing rule, let us assume that $C$ is a formula satisfying the conditions of the rules, then $C$ has a closed KEM-tree, let us call it $\mathcal{T}$. We show now that also the tree for $\Box_j^d C$ is closed.

$$
\begin{array}{c}
\mathsf{F}\Box_j^d C : w_0 \\
\mathsf{F}\Box_j^{d-1} C : (w_1^j, w_0) \\
\vdots \\
\mathsf{F}C : (w_d^j, \dots (w_1^j, w_0)) \\
\mathcal{T} : w_0' \\
\times
\end{array}
$$

All we have to do is to identify $w_0'$ with $(w_d^j, \dots (w_1^j, w_0))$ and we can repeat the proof for $C : w_0'$, in so far as $[(w_d^j, \dots (w_1^j, w_0)) | (w_d^j, \dots (w_1^j, w_0))]\sigma_{L_I^\delta}$.

For b) in the course of KEM-proofs, we generate labels according to the structure of the formulas involved, but, as we have already said, they also generate (counter)-models. The labels are intended to denote possible worlds and relations among them. Remeber (see section 4.1, and example 13): all the relevant information are recorded in the labels. So, to extract such information, we have to map labelled signed formulas to elements of fibred and dovetailed models. This is achieved with the help of three functions, namely $g$, $r$, and $f$. The function $g$ will map labels to sets of possible worlds: a singleton for constants, a set of worlds (possibly empty) for variables, and an actual world for auxiliary labels. The accessibility relation $R$ is assumed to be closed under specific conditions; but, we want to reconstruct it, through $r$, from the labels: path labels are intended to represent not only worlds, but also the chain of possible worlds leading to them. Finally, $f$, given an $LS$-formula, returns the evaluation of the formula with respect to the world(s) denoted by its label.

Let $\mathbf{m}^i = (S^i, R^i, a^i, h^i)$ be a model in $\mathcal{K}_i$ where: $S^i = \Phi_C^i$; $R^i$ is a binary relation on $S^i$; $a^i \in \Phi^A$, and $v^i$ is an evaluation function.

Let $g$ be a function from the set of $i$-pure labels occurring in a KEM-proof to $\wp(S^i)$ thus defined:

$$
g(t) = \begin{cases}
\{h(t)_t\} & \text{if } h(t) \in \Phi_C^i \\
\{w_n \in S^i \mid g(b(t)) R^i w_n\} & \text{if } h(t) \in \Phi_V^i \\
S^i & \text{if } t \in \Phi_V \\
\{a^i\} & \text{if } t \in \Phi^A
\end{cases}
$$

It may be possible that two labels have the same head, but they denote different worlds, this is way we have indexed $h(t)$ with the label itsself. However we shall drop the subscript, when this is harmless.

Let $r$ be a function from the set of $i$-pure labels occurring in a KEM-proof to $R^i$ thus defined:

$$
r(t) = \begin{cases}
\emptyset & \text{if } \ell(t) = 1 \\
g(t^1) R^i g(t^2), \dots, g(t^{n-1}) R^i g(h(t)) & \text{if } \ell(t) = n > 1
\end{cases}
$$

Let $f$ be a function from $LS$-formulas occurring in a KEM-proof to $v$ thus defined:

$$f(SA:t) \quad =_{def} \quad v(w_n^i, A) = 1 \text{ if } S = \mathsf{T}$$
$$f(SA:t) \quad =_{def} \quad v(w_n^i, A) = 0 \text{ if } S = \mathsf{F}$$

for all $w_n^i \in g(t)$.

Until now we have examined $i$-pure labels. Let $t$ be a not $i$-pure label. It can be decomposed into $i$-pure labels as follows: let $n \in \mathbb{N}$ such that $\forall m > n$, $s^m(t)$ is $i$-preferred. The label $c^n(t)$ is $i$-pure. A not $i$-pure label can be conceived as a recursive fibring of $i$-pure sub-labels.

Let $\mathbf{m}$ be a fibred structure, where the fibred function $\mathbf{F}$ is defined as follows:

$$\mathbf{F}_i(w_j) = g(w_0) \tag{7.7}$$

for each $w_n \in g(s^n(t))$, where $w_0 = h^1(c^n(t))$ and $c^n(t)$ is $i$-pure. Moreover we require that if $w_j \neq w_k$ then $\mathbf{F}_i(w_j) \neq \mathbf{F}_i(w_k)$. In the case of dovetailing we impose $\mathbf{F}_i(w_j) = w_j$

It is easy to see that $\mathbf{m}$ is a fibred or a dovetailed model for $L_I^\delta$.

**Lemma 23** *For any $t, s \in \Im$, if $[t|s]\sigma_{L_I^\delta}$ then $g(t) \cap g(s) \neq \emptyset$.*

**Proof** The proof is by induction on the number of applications of $\sigma_{L_i}$ in $\sigma_{L_I^\delta}$. First we have to prove the property for $\sigma_{L_i}$ and therefore for $\sigma^{A_1^i \cdots A_n^i}$. For a detailed proof see [1, 18].

The proof is by induction on the number of applications of $\sigma^{A_1^i \cdots A_n^i}$ in $\sigma_{L_i}$. We need first to prove the following:

**Lemma 24** *For any $t, s \in \Im$, if $[t|s]\sigma^{A_1^i \cdots A_n^i}$ then $g(t) \cap g(s) \neq \emptyset$.*

**Proof** We prove only the case for $DT4$, the other cases are similar and can be found in [1, 18]. The proof is by induction on the length of labels. If $\min\{\ell(t), \ell(s)\} = 1$, then at least one of $t$ and $s$ is either a constant or a variable, so that five cases will be present. By the definition of unifications $t, s$ are either: i) two constants, or ii) a variable and a constant, or iii) two variables, or iv) a variable and a label, or v) a constant and a label.[4]

Case i) Two constants unify if and only if they are the same constant, and so $t = s$; therefore from the definition of $g$, $g(t) = g(s)$ and so $g(t) \cap g(s) \neq \emptyset$.

Case ii) If $t$ (resp. $s$) is a variable and $s$ (resp. $t$) is a constant, then $g(t) = S^i$ and $g(s) \in \wp(S^i)$ therefore also in this case $g(t) \cap g(s) \neq \emptyset$.

[4]Cases ii), iii), and iv) are not found in KEM proofs, but they are useful both for dealing with cases in the inductive step and for case v).

Case iii) and iv) These cases are identical to the previous ones because: 1) $S^i$ is not empty, and 2) the variable is mapped to $S^i$ and the label to some world(s) in it.

Case v) This case implies that $[t|s]\sigma^T$. Let us assume, for the sake of economy, that $\ell(t) = 1$ and $\ell(s) = n > 1$. If $[t|s]\sigma^T$, then for each $h(s(s))$ such that $\ell(s(s)) > 1$ either $h(s(s)) \in \Phi_V$, or $h(s(s)) = t$; therefore $r(s) = tR^i h^2(s), \ldots, h^{n-1}(s)R^i h^n(s)$. If $h^2(s) \in \Phi_V$, then it denotes the set of worlds accessible from $t$; if $h^2(s) \in \Phi_C$, then $t = g(h^2(s))$,[5] in any case, through reflexivity $t \subseteq g(h^2(s))$, so we take $t$ as a representative of the set denoted by $h^2(s)$, which implies $tR^i h^3(s)$. We repeat the same argument until we arrive at $tR^i h^n(s)$: if $h^n(s) \in \Phi_C$, then $t = g(h^2(s))$ and so they denote the same world; if $h^n(s) \in \Phi_V$, then it denotes the set of worlds accessible from $t$; but $t$ belongs to such a set, therefore, in all cases $g(t) \cap g(s) \neq \emptyset$.

For the inductive step we have $\min\{\ell(t), \ell(s)\} = n > 1$. Let us assume inductively that the lemma is valid up to $n$; if $\ell(t) = \ell(s)$ we shall write $t$ and $s$ as $(h(t), b(t))$ and $(h(s), b(s))$, respectively. If $[t|s]\sigma^D$, by the definition of $\sigma^D$ we get $[b(t)|b(s)]\sigma^D$, for which the lemma holds; let $w_j$ be one of the worlds shared by $b(t)$ and $b(s)$, whence $w_j R^i h(t)$ and $w_j R^i h(s)$. We have now only to analyze what kind of labels are $h(t)$ and $h(s)$, which falls under the cases i), ii), and iii). Cases i) and ii) are the same as the inductive base. We have thus to examine case iii). Both $h(t)$ and $h(s)$ denotes the set of worlds accessible from $w_j$, but such a set is not empty because of the seriality of $R$.

If $\ell(t) \neq \ell(s)$, we shall assume that $\ell(t) < \ell(s)$ (the case $\ell(s) < \ell(t)$ is dealt with in the same way). If $[t|s]\sigma^T$ and $h(t) \in \Phi_C$ then $[t|s^{\ell(t)}(s)]\sigma^D$, therefore, combining the proofs of the previous case and case v) of the inductive base we obtain the desired result. If $h(t) \in \Phi_V$, then for all $s^n, n \leq \ell(t)$, $[h(t)|h(s)]\sigma = [h(t)|s^n]\sigma$ which means $g(t) \cap g(s^n(s)) \neq \emptyset$, and in particular $g(t) \cap g(s^{\ell(t)}(s)) \neq \emptyset$.

If $[t|s]\sigma^4$ then $h(t) \in \Phi_V$ and $[b(t)|s^{\ell(t)-1}(s)]\sigma^D$, for which the inductive hypothesis holds; let $w_j$ be such a shared world. $h(t)$ denotes all the worlds accessible from $w_j$, but, due to transitivity, the world(s) denoted by $h(s)$ belong(s) to $h(t)$ and so $g(t) \cap g(s) \neq \emptyset$.

This ends the proof of Lemma 24. ■

We return to the proof of the main lemma. If $\sigma_{L_i}$ consists of a single step of $\sigma^{A_1^i \cdots A_n^i}$, then $[t|s]\sigma_{L_i} = [t|s]\sigma^{A_1^i \cdots A_n^i}$; by Lemma 24 we obtain $g(t) \cap g(s) \neq \emptyset$.

Let us assume, inductively, that the lemma holds up to $n$. If $\sigma_{L_i}$ consists of $n + 1$ $\sigma^{A_1^i \cdots A_n^i}$-unifications, $[t|s]\sigma_{L_i} = [c^t(t)|c^s(s)]\sigma^{A_1^i \cdots A_n^i}$ where

[5]Due to the rules of KEM and the definition of the unifications, this case is possible only if $h^2(s)$ has been obtained by a previous unification, and so they do denote the same world.

$[s^t(t)|s^s(s)]\sigma_{L_i}$, which contains $n$ applications of $\sigma^{A_1^i \cdots A_n^i}$, and the lemma holds for it. We can now repeat the argument of Lemma 19 with respect to $[c^t(t)|c^s(s)]\sigma^{A_1^i \cdots A_n^i}$, proving thus that $g(t) \cap g(s) \neq \emptyset$. We have thus proved that the property holds also for $\sigma_{L_i}$. $\sigma_{L_I^\delta}$ results from the fibred combination of $\sigma_{L_i}$-unifications of $i$-pure labels. According to the interpretation of the fibred function given in 7.7 we can combine recursively the same argument as before.

This ends the proof of Lemma 23. ■

**Lemma 25** *For any $t, s \in \Im$, if $f(SA : t)$, $[t|s]\sigma_{L_I^\delta}$ then $f(SA : [t|s]\sigma_{L_I^\delta})$.*

**Proof** Let us suppose, by contradiction, that the lemma does not hold, so the proof trivially follows from Lemma 23, and the definition of $f$. ■

The $\alpha$-rules and *PB* are obviously sound rules in $\mathbf{m}$ in so far as they work locally. For the $\beta$-rules and *PNC*: by the hypothesis $[t|s]\sigma_{L_I^\delta}$, then, by Lemmas 23 and 25, there exists a world in $g([t|s]\sigma_{L_I^\delta})$, let us say $w_j$, where both $\beta$, $\beta_n$ $(n = 1, 2)$ and $\beta_{3-n}$ hold. This implies that $\beta_n$ and $\beta_n^C$ hold at $w_j$, which is a contradiction. The same argument can be applied in the case of *PNC*.

For the $\nu_i$-rules. Let us suppose $\nu_i = \mathsf{T}\Box_i A$ the case of $\nu_i = \mathsf{F}\Diamond_i A$ is identical. Let us suppose it does not hold, then for all $w_i \in g(t)$, $h(w_i, \Box_i A) = 1$ and for some $w_m \in g((s,t))$, $h(A, w_m) = 0$, where $(s,t)$ is $i$-preferred. If $t$ is $i$-preferred, then $h^i(w_i, \Box_i A) = 1$ implies $\forall w_j (w_i R^i w_j \rightarrow h^i(w_j, A) = 1)$, By the definitions of $g$ and $r$ we know $w_i R^i w_m$, obtaining thus a contradiction. If $t$ is not $i$-preferred, then $h^j(w_i, \Box_i A) = 1$ iff $h^i(\mathbf{F}_i(w_i), \Box_i A) = 1$. The label $(s,t)$ is not $i$-pure, but $(s, w_0)$, where $\mathbf{F}_i(w_i) = g(w_0)$, is. At this point we can repeat the same argument as before with $w_0$ instead of $w_i$. The proof for the $\pi_i$-rule is similar.

This ends the proof of Theorem 22. ■

## 5. FINAL REMARKS

In the last few years we have witnessed a luxuriant growth of multidimensional systems in every field of logic, although only few seeds have been sown in the garden of proof theory [4, 5, 7, 8].

In this work we have been mainly concerned with combinations of normal modal logics. It is easy to see that dovetailing of normal modal logics correspond to fusion [19]. However fibring is more general, and we have no constraints on the components. They may be as well combined logics. It's worth noting that, as far propositional normal modal logics are concerned, the system presented here satisfies all the conditions required for combined tableau calculi reported in [5]; in particular the condition stating "the labels that are part of tableau formulae represent words in models and do not contain other information". However KEM-based tableau calculi can be provided for different systems having Kripke-style semantics, e.g., quantified modal logics, non normal modal logics, conditional logics; but such systems require more information in the labels (see [1, 2]). The unification scheme we have proposed is not sensitive of such additional burden, and can be applied also for their combination, in so far as it is closely related to the fibring function.

We have not studied multi-modal logics with interaction axioms specifying how modalities relate each other. But in [3, 16, 17, 18] it has been shown how this formalism can be extended to handle various systems of multi-modal logic with interaction axioms. The key idea consists of providing either new substitutions and unifications or constraints on the already existing ones, both reflecting semantic properties of such axioms.

Let us consider the axiom $\Box_1 A \to \Box_2 A$; whose corresponding semantic condition is $R_2 \subseteq R_1$. This is achieved in KEM by defining a substitution as follows: $\sigma_1 : \Phi_V^1 \mapsto \Im^1 \cup \Im^2$. The above interaction axiom belongs to the family of inclusion axioms of the form $\langle a\rangle[b]A \to [c]\langle d\rangle A$, where $a$, $b$, $c$, and $d$ are strings of modalities obtained from composition and union (i.e., $\Box_1 \cdot \Box_2 A = \Box_1\Box_2 A$ and $\Box_1 \cup \Box_2 A = \Box_1 A \wedge \Box_2 A$). Such axioms characterize models satisfying $a, b, c, d$-incestuality, i.e., $\rho(a)^{-1} \cdot \rho(b) \subseteq \rho(c) \cdot \rho(d)^{-1}$, where $\rho$ is map from string of modalities to accessibility relation, see [6]. In more complex cases we may define an unification schema matching the appropriate $a, b, c, d$-incestuality condition, e.g., the first label ends with a string of world symbols corresponding to $a, b$ while the second corresponds to $c, d$; then we can apply (fibre) more unifications inside the strings of labels. For some interaction axioms (with a different form) special inference rules may be required (see [17]). Similarly non-normal modal logics can be dealt with by weakening the substitutions, e.g., $\sigma : \Phi_V \mapsto \Im^*$ where $\Im^* \subseteq \Im$. At the light of what we have already said the study of the proof theory of multi-modal logics reduces to the study of the components.

## Acknowledgments

Thanks to the anonymous referees for their valuable comments on an earlier version of this paper. This research was partially supported by the Australian Research Council under Large Grant No. A49803544.

## References

[1] Alberto Artosi, Paola Benassi, Guido Governatori, and Antonino Rotolo. Shakespearian modal logic: A labelled treatment of modal identity. In M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev (eds), *Ad-*

*vances in Modal Logic*, CSLI Publications, Stanford, 1998, 1–20.

[2] Alberto Artosi and Guido Governatori. A tableaux methodology for deontic conditional logics. In *ΔEON'98, 4th International Workshop on Deontic Logic in Computer Science*, CIRFID, Bologna, 1998: 65–84.

[3] Alberto Artosi, Guido Governatori, and Giovanni Sartor. Towards a computational treatment of deontic defeasibility. In Mark Brown and José Carmo (eds.), *Deontic Logic Agency and Normative Systems*, Springer-Verlag, Berlin, 1996: 27–46.

[4] Matteo Baldoni, Laura Giordano and Alberto Martelli. A tableau calculus for multimodal logics andl some (un)decidability results. In H. de Swart (ed.), *International Conference on Theorem Proving with Analytic Tableaux and Related Methods*, Springer-Verlag, Berlin, 1998: 44–59.

[5] Bernhard Beckert and Dov M. Gabbay. Fibring semantic tableaux. In H. de Swart (ed.), *International Conference on Theorem Proving with Analytic Tableaux and Related Methods*, Springer-Verlag, Berlin, 1998: 77–92.

[6] Laurent Catach. Normal multimodal logics. In *Proc. Nat. Conf. on AI (AAAI'88)*, 491–495, 1988.

[7] Marcello D'Agostino and Dov M. Gabbay. Fibred tableaux for multi-implication logics. In P. Miglioli, U. Moscato, D. Mundici, and M. Ornaghi, (eds.), *Theorem Proving with Analytic Tableaux and Related Methods*, Springer-Verlag, Berlin, 1996: 16–35.

[8] Marcello D'Agostino, Dov M. Gabbay, and Alessandra Russo. Grafting modalities onto substructural implication systems. *Studia Logica* 59: 65–102, 1997.

[9] Marcello D'Agostino and Marco Mondadori. The taming of the cut. *Journal of Logic and Computation*, 4: 285–319, 1994.

[10] Melvin Fitting. *Proof Methods for Modal and Intuitionistic Logics*. Reidel, Dordrecht, 1983.

[11] Dov M. Gabbay. Fibred semantics and the weaving of logic I: Modal and intuitionistic logics. *Journal of Symbolic Logic*, 61: 1057–1120, 1996.

[12] Dov M. Gabbay. *Labelled Deductive System*. Oxford University Press, 1996.

[13] Dov M. Gabbay. An overview of fibred semantics and the combination of logic. In F. Baader and K. Schulz, (eds.), *Frontiers of Combining Systems*, Kluwer, Dordrecht, 1996: 1–55.

[14] Dov M. Gabbay. *Fibring Logic*. Oxford University Press, 1998.

[15] Olivier Gasquet. Optimization of deduction for multimodal logics. In Masuch, Marx and Plós (eds.), *Applied Logic: What, How and Why?*. Kluwer, Dordrecht, 1993.

[16] Guido Governatori. Labelled tableaux for multi-modal logics. In P. Baumgartner, R. Hähnle and J. Posegga (eds.) *Theorem Proving with Analytic Tableaux and Related Methods*, Springer-Verlag, Berlin, 1995: 79–94.

[17] Guido Governatori. Labelling ideality and subideality. In D. Gabbay, H. J. Ohlbach (eds.), *Practical Reasoning*, Springer-Verlag, Berlin, 1996: 291–304.

[18] Guido Governatori. Un modello formale per il ragionamento giuridico. PhD. Thesis, University of Bologna, 1997.

[19] Marcus Kracht and Frank Wolter. Properties of indipendently axiomatizable bimodal logics. *Journal of Symbolic Logic*, 56:1485–1991, 1991.

[20] Hans Jürgen Ohlbach. Semantics-based translation methods for modal logics. *Journal of Logic and Computation*, 1:691–746, 1990.

[21] Hans Jürgen Ohlbach. Optimized translation of multi modal logic into predicate Logic. A. Voronkov (ed.) *Proc. of Logic Programming and Automated Reasoning (LPAR)*, Springer-Verlag, Berlin, 253–264, 1993.