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Dealing with Label Dependent Deontic Modalities

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Abstract. In this paper, following Scott's advice, we argue that normative reasoning can be represented in a multi-setting framework; in particular in a multi-modal one, where modalities are indexed. Indexed modalities can model several aspects involved in normative reasoning. Systems are combined using Gabbay's fibring methodology which provides complete semantics that can be used to model a labelled tableau-like proofs system.

Introduction

It is widely recognized that not all norms have the same status; in general, norms arise from different sources, for example we can have constitutional laws, ordinary laws, regional laws, municipal regulations and so on. Norms can also have different ranges of applicability, e.g., civil laws, penal laws, martial laws, ... Some norms concern facts whereas others are about norms themselves. Moreover there is no agreement on what counts as a norm and what does not. The only *trait d'union* among them is that they are concerned with the notions of obligation and permission (or similar notions). Consequently, deontic logic, in which these notions are given a modal representation, has been proposed as a logical framework for formalising norms. However, deontic logic cannot account in a natural way for the more fine grained distinction between norms alluded to above. Therefore we simply propose to represent the more fine tuned information using labels, attaching a label to each deontic operator to distinguish and interpret these different concepts of norm.

Usually, normative reasoning deals with several intensional notions at once, for example, knowledge, necessity, obligatory, temporal notions and so on. An example of a norm involving different intensional notions is article 368 of the Italian Code of Penal Law concerning slander, stating

> Whoever brings a legal action addressed to the competent authority or to another authority which has to report to the first one, against a third party known by the plaintiff to be innocent, i.e., he/she brings false evidence against the third party, should be jailed from a minimum of two to a maximum of six years.

This example involves four agents: the plaintiff, the competent authority, the other authority, which plays the twofold role of authority and addressee, and the third party. For the first three agents we ascribe a notion of knowledge. We also have two notions of obligation, the first arises from the norm itself, and the second is that referred to inside the norm. Following Scott's advice (Scott, 1970) intensional notions can be represented in a multi-modal setting by assigning a different modal operator to each of them. This leads straightforwardly to the notion of indexed modality, where the indices belong to the set of intensional notions related to the phenomena under investigation. Indexing modalities with elements of a given set might produce useful applications in the field of normative systems. For example, when the elements correspond to instants of time the resulting logic corresponds to a temporal deontic logic[1]. When the elements are agents the corresponding logic may model a logic of authorities and addressees[2]. When the indices themselves are formulas, the resulting logic is a context logic similar to conditional logic[3]. When indices represent actions we have dynamic logic[4].

Until now we have not had conditions on the set of indices. Let us suppose that the set of indices is ordered and we want to maintain its structure; in such a case we have to impose conditions on the combination of labelled logics defining a hierarchical composition (or embedding) of the labelled logics. This means that we consider only formulas where an occurrence of modality cannot lie in the scope of a modality indexed with a label of lower degree. It is worth noting that the labels may not be homogeneous: one label might be a temporal pointer whereas another is interpreted deontically. For example let $I = \{t, d\}$ be the set of labels with $d \prec t$ where t is a temporal label and d is a deontic label. According to such an interpretation we have two distinct sets of modalities, $\Box_t^{n_t}$ and $\Box_d^{n_d}$, which give raise to two different (multi-)modal logics, a temporal logic for the former (e.g., the logic of Since and Until) and a deontic logic (e.g., the Jones and Pörn (1985,1986) logic of ideality and sub-ideality) for the latter; in such a case we obtain a temporalization of the deontic operators (Finger & Gabbay, 1993). In this perspective we have two different logics combined together where we distinguish between internal and external logics; in the previous example the internal logic (the deontic one) is concerned with the deontic state of affairs, while the external takes care of the "history of deontic systems", expressing whether a norm is valid in a span of time, or how long a norm has held. Moreover it is possible to add another temporal dimension, inside the deontic one. This third logic may be conceived of as the logic modelling the temporal constraints imposed by the norms.

What happens when we combine though an embedding two deontic logics? The resulting logic can be used to describe sources of law or different degrees of adjudication. In fact norms and normative systems are usually arranged according to hierarchies. Existing approaches to normative hierarchies have studied the problem with respect to

- authorities and addressees;
- structure of the hierarchies;
- iteration of the same modality.

But the question of the logical formalization of norms of different strength in a multi-modal framework has not been raised. It is also possible to distinguish between "primary" norms (norms about facts) and "secondary" norms (norms about norms).[5] We

[1]See (Chellas, 1980, Thomason, 1981, van Eck, 1981) for interactions between temporal and deontic notions.

[2]The question of authorities and addressees is examined in (Bailache, 1991, Krogh & Herrestad, 1996).

[3]Several works have proposed to read the conditional $A > B$ as a unary operator dependent on the antecedent, see for example (Åqvist, 1985, Chellas, 1980, Lewis, 1986).

[4]Several works have proposed dynamic logics for formalising normative reasoning, see (Dignum et al., 1994, Meyer, 1987, Meyer, 1988).

[5]The terminology of primary and secondary norms is not unanimous; it varies according to philo-

believe that the embedding of deontic logics may prove useful for the analysis of the above problems.

In Section 1 we shall present two classes of Labelled Modal Logics (LML) we believe can be used to formalize normative reasoning; in Section 2 we overview a general semantic methodology for combining systems, and we state conditions under which completeness of such systems obtain; in Section 3 we shall investigate combined systems from a computational point of view using the tableau-like proof system KEM.

1 Labelled Modal Logics

In this section we introduce the language of Labelled Modal Logics, and we provide fibred (dovetailed) semantics (Gabbay, 1996b, 1996c) for them.

The language of a Labelled Modal Logic is defined by:

- a set of labels (or indices) I;
- a set of propositional letters $\{P_1, P_2, \ldots\}$;
- boolean operators $\wedge$, $\vee$, $\rightarrow$, $\neg$;
- A set of intensional operators $\{\Box_p : p \in I\}$

The set of well formed formulas is then defined inductively in the usual way. For the sake of simplicity, in order to provide an overview of the methodology, we shall confine ourselves to combination of the normal modal logic arising from the combination of the axioms $D, T, 4, B, 5$ and the axioms obtained by prefixing n $\Box$ to them; however there is no constraint on the systems that can be combined. Therefore a LML is the smallest set of formula containing the following axioms

- classical propositional tautologies
- $\Box_p(A \rightarrow B) \rightarrow (\Box_p A \rightarrow \Box_p B)$ $p \in I$

and closed under Modus Ponens and Necessitation

(Nec) $\dfrac{A}{\Box_p A}$ $p \in I$

Notice that LMLs correspond to normal multi-modal logics without interaction axioms. However we can use any logic as a component of a LML.

Example 1 In this framework we can combine two modal logics, e.g., an epistemic $K45$-modality and a deontic KD-modality; the resulting logic, let us call it ED, is able to formalize "knowledge of obligations", and "obligation of knowledge". For example we can ascribe to

$$\mathcal{B}OA \rightarrow \mathcal{K}\mathcal{B}PA$$

"if an agent believes ($\mathcal{B}$) that something is obligatory (OA) then she knows ($\mathcal{K}$) that she believes ($\mathcal{B}$) it is permitted (PA) ", and

$$\mathcal{K}OA \rightarrow \mathcal{K}PA$$

"if an agent knows that something is obligatory then she knows it is permitted" as formulas representing knowledge of obligations. On the other hand, examples of obligation of knowledge are the infamous Miranda Warnings, that (roughly) can be formalized as

$$Arrested(x) \rightarrow O\mathcal{K}_x\big(\bigwedge Rights(x)\big) \ ,$$

and the already mentioned article 368 of the Italian Code of Penal Law.

sophical beliefs of scholars.

1.1 Hierarchical Modal Logic

In this section we shall be concerned with a particular class of set, i.e., the class of (totally) ordered labels, i.e., the labels belong to a structure of the form $(I, \prec)$.

To define the set of well formed formulas ($WFF_{I_\prec}$) we first need to define the modal nesting of a formula.

DEFINITION 2

- *No atomic formula is p-modalized;*
- $\Box_p A$ *and* $\Diamond_p A$ *are p-modalized;*
- *if* A, B *are p-modalized, so are* $\neg A$, $A \wedge B$, $A \vee B$ *and* $A \rightarrow B$*;*
- *if* A *is p-modalized and* B *is not q-modalized for* $p \prec q$*, then* $A \wedge B$, $A \vee B$ *and* $A \rightarrow B$ *are p-modalized.*

The set of well formed formulas $WWF_{I_\prec}$ is then defined as:

DEFINITION 3

- *if* A *is atomic then* $A \in WWF_{I_\prec}$*;*
- *if* $A, B \in WWF_{I_\prec}$ *then* $\neg A, A \wedge B, A \vee B, A \rightarrow B \in WWF_{I_\prec}$*;*
- *if* $A \in WWF_{I_\prec}$ *is not p-modalized then* $\Box_q A \in WWF_{I_\prec}$ *for* $q \prec p$*.*

A Hierarchical Modal Logic (HML) is the smallest set of formula containing the following axioms:

- classical propositional logic;
- $\Box_p(A \rightarrow B) \rightarrow (\Box_p A \rightarrow \Box_p B)$, if A and B are not q-modalized for $p \prec q$;

and closed under Modus Ponens and Hierarchical necessitation

$$(Nec_{I_\prec}) \quad \frac{A}{\Box_p A}[\text{if } A \text{ is not } q\text{-modalized for } p \prec q]$$

A Hierarchical Deontic Logic is a logic where a low level deontic operator cannot be applied to formulas containing occurrences of an higher level operator. Higher level norms may impose constraints on lower level norms (but usually not vice versa), and norms at different levels might obey different principles.

We can extend such a logic with various modal principles for the different modal operators. In this way we can combine hierarchically several modal logics. We shall use the sequence made of the modal logic names as the name of the resulting hierarchical logic. For example $T, D, K4$ is the hierarchic modal logic resulting from applying a modality of type T to a modality of type $D, K4$, where $D, K4$ is a serial modality applied to a transitive one. We shall use $[L]$ and $\langle L \rangle$ to denote, respectively, the $\Box$ and $\Diamond$ operators of the modal logic L.

EXAMPLE 4 a theorem of such a logic is

$$[T][D][K4]A \rightarrow \langle D \rangle [K4][K4]A$$

In fact it can be proved as follows

1	$[K4]A \to [K4][K4]A$	Axiom 4
2	$[D]([K4]A \to [K4][K4]A)$	$Nec_{I_{\prec}}$
3	$[D][K4]A \to [D][K4][K4]A$	Axiom $K_{[D]}$
4	$[D][K4][K4]A \to \langle D\rangle[K4][K4]A$	Axiom D
5	$[D][K4]A \to \langle D\rangle[K4][K4]A$	PC
6	$[T]([D][K4]A \to \langle D\rangle[K4][K4]A)$	$Nec_{I_{\prec}}$
7	$[T][D][K4]A \to [T]\langle D\rangle[K4][K4]A$	Axiom $K_{[T]}$
8	$[T]\langle D\rangle[K4][K4]A \to \langle D\rangle[K4][K4]A$	Axiom T
9	$[T][D][K4]A \to \langle D\rangle[K4][K4]A$	

In example 1 we have proposed the logic ED as a logic formalising obligations of knowledge and knowledge of obligations, but in such a framework the internal and external notion of knowledge are the same. However we can conceive situations where the two notions of knowledge behave differently and they are different kinds of knowledge with different fields of applications, (knowledge of law) and (knowledge of fact). This kind of notions can be easily represented in HML. Another interesting application of HML in the field of deontic logic has been proposed by Cholvy and Cuppens (1998), who propose an HML to describe a system able to deal with conflicting codes.

Example 5 To emphasize the rôle of HML in normative reasoning we propose two systems arising from the combination of three modalities N_1, N_2, N_3 such that $3 \prec 2 \prec 1$. In the first case they represent, respectively, constitutional laws, national laws, and regional lows. Usually constitutional norms are not about facts, they rule principles other norms should obey. This is the reason why we can assume a KT system for N_1. On the other hand the logics of N_2 and N_3 are D systems. Let us suppose that there is a constitutional norm stating that a given state of affairs should be regulated. This can be expressed as follows

(3) $N_1(A \to N_2B)$

Moreover it is possible, indeed this is often the case in real life, that the national authority demands the actuation of the norm to local authorities according to the actual local context. So $B = C \to N_3D$, therefore 3 turns out to be

(4) $N_1(A \to N_2(C \to N_3D))$

Another reading of the modalities which would be worth investigating is one where they are interpreted over degrees of adjudication.

The main issue of both cases consists in the principle that higher norms can impose constraints on lower ones, but not the other way around. Similarly lower levels of adjudication can be (over)ruled by higher ones.

2 Combining Modal Logics

In the previous sections we introduced two simple classes of combined logics (LML and HML), and we argued for their applicability to the field of normative reasoning. In this section we present a general semantic methodology, called fibring, for combining modal (deontic) logics, and we state conditions under which completeness of combined systems obtain. It will be then easy to see that semantics for LML and HML can be given in terms of fibred (dovetailed) models.

The methodology of fibring allows us to combine arbitrary logical systems to form a new system in a uniform way, "fibring" their models (for detailed expositions see Gabbay, 1996a, 1996b, 1996c, 1998). The main idea of fibring is very simple, and provides a new concept of possible world semantics. Let us suppose we want to combine two modalities $\Box_1$ and $\Box_2$ characterised, respectively, by the classes of models $\mathcal{K}_1$ and $\mathcal{K}_2$. We know how to evaluate $\Box_1 A$ in $\mathcal{K}_1$, $\Box_2 A$ in $\mathcal{K}_2$ and propositional formulas in both. All we need is a method for evaluating $\Box_1$ (resp. $\Box_2$) with respect to $\mathcal{K}_2$ (resp. $\mathcal{K}_1$). Each time we have to evaluate a formula $\mathcal{A}$ of the form $\Box_2 A$ in a world in a model of $\mathcal{K}_1$ we associate, via the fibring function $\mathbf{F}$, to the world a model in $\mathcal{K}_2$ where we calculate the truth value of the formula. Formally

$$w \vDash_{\mathbf{m}\in\mathcal{K}_1} \Box_2 A \iff \mathbf{F}_{\mathbf{m}}(w) \vDash_{\mathbf{m}'\in\mathcal{K}_2} \Box_2 A$$

$\mathcal{A}$ holds in w iff it holds the model associated to W through the fibring function $\mathbf{F}$. But now we are in an appropriate model for evaluating $\mathcal{A}$.

In the next two sections we define two ways, fibring and dovetailing, in which semantics can be combined. We first explain the concepts by taking a simple example. Suppose we want to combine two modal logics L_1 and L_2. Let $\mathcal{K}_1, \mathcal{K}_2$ be the respective Kripke semantics of the logics. Let $\mathbf{m}$ be a model in $\mathcal{K}_1$ and let t be a possible world of $\mathbf{m}$. The semantic construction which combines the logics associates a model $\mathbf{n}$ in $\mathcal{K}_2$ with t. The different methodologies of combination differ on the kind of model $\mathbf{n}$ that we use. For *fibring* logics, we require that $\mathbf{n}$ be any model in $\mathcal{K}_2$. For *dovetailing*, we require that $\mathbf{n}$ be a model of $\mathcal{K}_2$ such that for any *atomic* P, $t \vDash P$ iff $\mathbf{n} \vDash P$ (i.e., the fibred model must agree with the values t gives to atoms).

2.1 Language of combined logics

Let $\mathbb{L}_p$ be the language of the logic L_p. The expressions E^p of the language $\mathbb{L}_p$ are built up using $\mathbb{L}_p$-constructors (connectives and operators) from a set of atomic units P^p. We schematically write $E^p(P^p_1, \ldots, P^p_n)$ to indicate that E^p is built up from the atoms $P^p_1, \ldots, P^p_n \in P^p$. We now define the fibred language $\mathbb{L}_{(x_1,\ldots,x_n)}$ as follows:

- Let $\mathbb{L}_{(p)}$ be $\mathbb{L}_p$, $p \in I$;
- Let $\bar{y}$ be $(q, y_1, \ldots, y_k)$, $q, y_1, \ldots, y_k \in I$ and $p \neq q$, or, for hierarchical combination, $q \prec p$. Let $\mathbb{L}_{(p)*\bar{y}}$ be the family of all expressions of the form $\alpha = E^p(P_1/A_1, \ldots, P_n/A_n)$ where $E^p(P_1, \ldots, P_n) \in \mathbb{L}_p$ and $A_1, \ldots, A_n$ are in $\mathbb{L}_{\bar{y}}$, and P_j/A_j indicates the substitution of of A_j to P_j in E^p;
- $\mathbb{L}_I = \bigcup_{\bar{y}} \mathbb{L}_{\bar{y}}$.

Let L_1 and L_2 be two modal (deontic) logics whose respective languages are $\mathbb{L}_1$ and $\mathbb{L}_2$. The language $\mathbb{L}_{1,2}$ of $L_{1,2}$ contains $\mathbb{L}_1$ and $\mathbb{L}_2$ as well as all the expression obtained from them by substituting atoms with formulas of the other language. Thus, for example, $\mathbb{L}_{(1,2)}$ contains $\Box_1(P \wedge \Diamond_1(Q \vee \Box_2 P))$. $\Box_1(P \wedge \Diamond_1 Q)$ is in $\mathbb{L}_{(1)}$ and $Q \vee \Box_2 P$ is in $\mathbb{L}_{(2)}$, thus we can replace Q with $Q \vee \Box_2 P$ obtaining the desired result.

2.2 Fibring of modal logics

Let $L_p, p \in I$ be modal logics in the respective language $\mathbb{L}_p$, with $\Box_p, p \in I$, respectively. Let $\mathcal{K}_p$ be a class of models $\{\mathbf{m}^p_1, \mathbf{m}^p_2, \ldots\}$ for which L_p is complete. Each model $\mathbf{m}^p_n$ has the form (S, R, a, h) where S is the set of possible worlds, $a \in S$ is the actual world and $R \subseteq S^2$ is the accessibility relation. h is the assignment function, a binary function, giving a value $h(t, P) \in \{0, 1\}$ for any $t \in S$ and atomic P. The actual world a plays a role in the semantic evaluation in the model, in so far as satisfaction in the model is defined as satisfaction at a. We can assume that the models satisfy the following condition:

$$S = \{x \mid \exists n \;\; aR^n x\} \ .$$

This assumption does not affect satisfaction in models because points not accessible from a by any power R^n of R do not affect truth values at a. Moreover we assume that all sets of possible worlds in any $\mathcal{K}_p$ are all pairwise disjoint, and that there are infinitely many isomorphic (but disjoint) copies of each model in $\mathcal{K}_p$. We use the notation $\mathbf{m}$ for a model and present it as $\mathbf{m} = (S^{\mathbf{m}}, R^{\mathbf{m}}, a^{\mathbf{m}}, h^{\mathbf{m}})$ and write $\mathbf{m} \in \mathcal{K}_p$, when the model $\mathbf{m}$ is in the semantics $\mathcal{K}_p$. Thus our assumption boils down to $\mathbf{m} \neq \mathbf{n} \Rightarrow S^{\mathbf{m}} \cap S^{\mathbf{n}} = \varnothing$. In fact a model can be identified by its actual world, i.e., $\mathbf{m} = \mathbf{n}$ iff $a^{\mathbf{m}} = a^{\mathbf{n}}$.

Henceforth we shall use $\lneqq$ as a notation for $\neq$ referring to LML and $\prec$ referring to HML.

Definition 6 *A fibred model is a structure*

$$(W, W_{p,p\in I}, W_a, R, w_0, h, \mathbf{F})$$

where $W = \bigcup_{\mathbf{m}\in\cup_p\mathcal{K}_p} S^{\mathbf{m}}$; $W_{p,p\in I} = \{a^{\mathbf{m}} \mid \mathbf{m} \in \mathcal{K}_p\}$; $W_a = \bigcup_p W_p$; $R = \bigcup_{\mathbf{m}\in\cup_p\mathcal{K}_p} R^{\mathbf{m}}$; $w_0 \in W_a$ *is the actual world, for hierarchical models we further require that* $w_0 \in W_p$ *such that* $p = \max(I)$; $h(t, q) = h^{\mathbf{m}}(t, q)$, *for the unique* $\mathbf{m}$ *such that* $t \in S^{\mathbf{m}}$; $\mathbf{F} : I \times W \mapsto W_p$, *is the fibring function. The fibring function* $\mathbf{F}$ *is a function giving for each* i *and each* $w \in W$ *another point (actual world) in* W_i *as follows:*

$$\mathbf{F}_p(w) = \begin{cases} w & \textit{if } w \in S^{\mathbf{m}} \textit{ and } \mathbf{m} \in \mathcal{K}_p \\ \textit{a value in } W_p, & \textit{otherwise} \end{cases}$$

such that if $x \neq y$ *then* $\mathbf{F}_p(x) \neq \mathbf{F}_p(y)$.

Satisfaction is defined as follows with the usual truth tables for boolean connectives:

$$\begin{array}{lll} t \vDash P & \textit{iff} & h(t, P) = 1 \\ t \vDash \Box_p A & \textit{iff} & \begin{cases} t \in \mathbf{m}^p \textit{ and } \forall s(tRs \to s \vDash A) \\ t \in \mathbf{m}^q,\ p \lneqq q \textit{ and } \mathbf{F}_p(t) \vDash \Box_p A \end{cases} \end{array}$$

We say the model satisfies A *iff* $w_0 \vDash A$.

Example 7 Before stating the conditions under which an LML (HML) is complete with respect to fibred (hierarchical) models we show how to evaluate $\mathcal{A} = \neg([T][D][K4]P \to \langle D\rangle[K4][K4]P)$ in a fibred hierarchical model for $T, D, K4$. First of all we define the appropriate model. Let $\mathbf{m}^T = (W^T, R^T, w^T, h^T)$, $\mathbf{m}^D = (W^D, R^D, w^D, h^D)$, $\mathbf{m}^{K4} = (W^{K4}, R^{K4}, w^{K4}, h^{K4})$ be, respectively, a reflexive, a serial, and a transitive models; then the resulting hierarchical model HML is $\mathbf{m}_{HML} = (W, W_p, W_a, R, w_0, \mathbf{F}, h)$, where $W = W^T \cup W^D \cup W^{K4}$, $W_p = \{w^T\}, \{w^D\}, \{w^{K4}\}$,

$W_a = \{w^T, w^D, w^{K4}\}$, $R = R^T \cup R^D \cup R^{K4}$, and $w_0 = w^T$.

$$\begin{aligned}
\models_{\text{HML}} \mathcal{A} &\iff w_0 \models_{\text{HML}} \mathcal{A} \iff w_0 \models_T \mathcal{A} \\
&\iff w_0 \not\models_T [T][D][K4]P \to \langle D\rangle[K4][K4]P \\
&\iff \begin{cases} (i) \;\; w_0 \models_T [T][D][K4]P & \text{and} \\ (ii) \; w_0 \not\models_T \langle D\rangle[K4][K4]P \end{cases} \\
(i) &\iff \forall w_t \in W^T : w_0 R^T w_t, w_t \models_T [D][K4]P
\end{aligned}$$

At this point we have to evaluate a formula of $\mathbb{L}_{(D,K4)}$ in a world in a T-model, thus we have to apply the fibring function to move to an appropriate D-model

$$\begin{aligned}
&\iff \mathbf{F}_D(w_t) \models_D [D][K4]P \\
&\iff \forall w_d \in W^D : \mathbf{F}_D(w_t) R^D w_d, w_d \models_D [K4]P
\end{aligned}$$

We have now to repeat the above reasoning with respect to w_d and $K4$

$$\begin{aligned}
&\iff \mathbf{F}_{K4}(w_d) \models_{K4} [K4]P \\
&\iff \forall w_{k4} \in W^{K4} : \mathbf{F}_{K4}(w_d) R^D w_{k4}, w_{k4} \models_{K4} P
\end{aligned}$$

We return now to the evaluation of (ii), where we have to apply the fibring function

$$\begin{aligned}
(ii) &\iff \mathbf{F}_D(w_0) \not\models_D \langle D\rangle[K4][K4]P \\
&\iff \forall w'_d \in W^D : \mathbf{F}_D(w_0) R^D w'_d \not\models_D [K4][K4]P \\
&\iff \mathbf{F}_{K4}(w'_d) \not\models_{K4} [K4][K4]P \\
&\iff \exists w'_{k4} \in W^{K4} : \mathbf{F}_{K4}(w'_d) R^{K4} w'_{k4}, w'_{k4} \not\models_{K4} [K4]P \\
&\iff \exists w''_{k4} \in W^{K4} : w'_{k4} R^{K4} w''_{k4}, w''_{k4} \not\models_{K4} P \\
&\iff w''_{k4} \models_{K4} \neg P
\end{aligned}$$

Since R^T is reflexive $w_0 R^T w_0$ holds, so, without any loss of generality we can replace w_t with w_0, and so we can identify $\mathbf{F}_D(w_t)$ with $\mathbf{F}_D(w_0)$. According to the conditions on $\mathbf{F}$ and $\mathbf{m}_{HML}$, $\mathbf{F}_D(w_0) = w^D$. R^D is serial. This ensures that the set of worlds accessible from w^D is not empty. Let w_d be in example 4 that $[T][D][K4]P \to \langle D\rangle[K4][K4]P$ is a theorem of $T, D, K4$.

THEOREM 8 (Completeness theorem for the fibred logic L_I^F) *Let $L_p, p \in I$ be modal logics in the respective language $\mathbb{L}_p$ with classes of structures $\mathcal{K}_p$ and set of theorems T_p (i.e., $T_p = \{A \text{ of } \mathbb{L}_p \mid A \text{ is valid in all } \mathcal{K}_p \text{ models }\}$). We consider two options for fibring. The general option where the logics L_p are assumed to have pairwise disjoint sets of atomic propositions (referred to as disjoint fibring) and the ordinary fibring where all L_p share the same set of atoms. Let T_I^F be the following set of wffs of L_I^F.*

1. $T_p \subseteq T_I^F$, for every $p \in I$.

For the case of ordinary fibring of n logics, we need axioms 1a and 1b belsow. If we are fibring an infinite number of logics, 1a is not needed.

1a. If A is a Boolean combination of atoms and $\alpha_{\bar{y}_n}$ is in the $\bar{y}$-th fibred language, then $A \to \alpha_{\bar{y}_n} \in T_{\bar{y}}$ implies $A \to \bigvee_n \alpha_{\bar{y}_n} \in T_I^F$, where in $\alpha_{\bar{y}_n}$ every atom is in the scope of a modality.

1b. If $A(x_j) \in T_p$ then $A(x_j/\Box_q\alpha_j) \in T_I^F$, for any $\Box_q\alpha_j \in \mathbb{L}_q, q \in I$; for HML we further require $q \prec p$.

2. **Modal Fibring Rule:**[6] *If $\Box_p$ is the modality of L_p and $\Box_q$ that of L_q, where p, q are arbitrary, with $p \gneqq q$ and*

$$C = \bigwedge_{k=1}^{n} \Box_p A_k \to \bigvee_{k=1}^{m} \Box_p B_k \in T_I^F$$

then for all d, $\Box_q^d C \in T_I^F$.

3. T_I^F is the smallest set closed under 1, 2, modus ponens and substitution.

Then T_I^F is the set of all wffs of L_I^F valid in all the fibred structures of L_I^F.

Proof. See (Gabbay, 1996a, 1998). Notice that (Gabbay, 1996a) deals with the case that the logics $L_p, p \in I$ have no atoms in common. ■

2.2.1 Dovetailing of modal logic

Dovetailing arises in many applications where the fibred model at world t has the world t itself as its actual world. The notions of dovetailing results from that of fibring when for all $p \in I$ and for all atomic P

$$h(t, P) = h(\mathbf{F}_p(t), P) .$$

Accordingly we can identify the actual world of the model fibred at t, $\mathbf{F}_p(t)$, with t. The fibring function $\mathbf{F}$ is no longer needed, since we identified t with $\mathbf{F}_p(t)$.

Let $L_p, p \in I$ be modal logics as in section 2.2, with $\mathcal{K}_p$ the class of models for L_p. Let L_I^D (the dovetailing combination of $L_p, p \in I$) be defined semantically through the class of all (dovetailed) models of the form (W, R, a, h), where W is a set of worlds, $a \in W$, h is an assignment as before, and for each $p \in I, R(p) \subseteq W \times W$. We require that for each p, $(W, R(p), a, h)$ is a model in $\mathcal{K}_p$. We further require the following:
Let $t \in W$ be such that there exist $n_1, \ldots, n_k$ and $p_1, \ldots, p_k$ such that $aR^{n_1}(p_1) \circ R^{n_2}(p_2) \ldots \circ R^{n_k}(p_k)t$ holds; for HML, we require also $p_1 \succ p_2 \succ \ldots \succ p_n$.

We define the notion of $w \vDash A$ by induction.

- $w \vDash P$ if $h(w, P) = 1$ for P atomic.
- $w \vDash \Box_p A$ if for all $y \in W$, such that $wR(p)y$ we have $y \vDash A$.
- $\vDash A$ iff for all models and actual worlds $a \vDash A$.

Theorem 9 (Completeness theorem for the dovetailed logic L_I^D) *Let $L_p, p \in I$ be modal logics with semantical classes of structures $\mathcal{K}_p$ and set of theorems T_p. Let T_I^D be the following set of wffs of L_I^D.*

1. $T_p \subseteq T_I^D$

[6] Intuitively the meaning of the modal fibring rule has to do with substitutions of wffs of one language into a formula of the other language. If the substituted wffs are related (proof theoretically) we want to propagate this relation into the other language.

There are formal similarities between the modal fibring rule and necessitation. Consider a formula C built up only from 'atoms' of the form $\Box_p B_k$. Then our special rule of necessitation says that from $\vdash C$ we can deduce $\vdash \Box_q^d C$ for any modality $\Box_q$ *other than* $\Box_p$. Notice that in presence of (hierarchical) necessitation the modal fibring rule is a derived rule.

2. **Modal Dovetailing Rule**:[7] *If* $\Box_p$ *is the modality of* L_p *and* $\Box_q$ *that of* L_q*, where* p, q *are arbitrary, with* $p \lneqq q$ *and*

$$C = \bigwedge_{k=1}^{n} \Box_p A_k \wedge \bigwedge_{k=1}^{m} \Diamond_p \neg B_k \to \bigvee_{k=1}^{r} P_k \in T_I^D$$

then for all d $\Box_q^d C \in T_I^D$*. Where* P_k *are atoms or their negations, and* $P_1, \ldots, P_r$ *list all the atoms or their negations appearing in any* A_k *or* $B_k, k = 1, 2, \ldots$.

3. T_I^D *is the smallest set closed under 1, 2, modus ponens and substitution.*

Then T_I^D *is the set of all wffs of* L_I^D *valid in all the dovetailed structures of* L_I^D*.*

Proof. See (Gabbay, 1996a). ■

Theorem 10 *Assume* $L_p, i \in I$ *all are extensions of* K *formulated using traditional Hilbert axioms and the rule of (hierarchical) necessitation, then* L_I^D *(the dovetailing of* L_p*) can be axiomatised by taking the union of the axioms and the rules of (hierarchical) necessitation for each modality* $\Box_p$ *of each* L_p

Proof. See (Gabbay, 1996a). ■

From the above theorems it follows that LML and HML are complete with respect to the fibred and dovetailed semantics.

Remark 11 In this work we take into account only combinations of logics without interaction axioms. However the fibring methodology can be used also in such a case. For example, completeness can be proved for dovetailed systems arising from interaction axioms of the form $\langle a \rangle [b] A \to [c] \langle d \rangle A$, where a, b, c and d are strings of modalities obtained from composition and union (i.e., $\Box_1 \cdot \Box_2 A = \Box_1 \Box_2 A$ and $\Box_1 \cup \Box_2 A = \Box_1 A \wedge \Box_2 A$). Such axioms characterize models satisfying a, b, c, d-incestuality, i.e. $\rho(a)^{-1} \cdot \rho(c) \subseteq \rho(b) \cdot \rho(d)^{-1}$, where ρ is a map from strings of modalities to accessibility relation, see (Catach, 1988). On the other hand we believe that the methodology we propose can be used to determine characterization of interaction axioms.

In the next section we provide a tableau-like proof method for LML and HML, and combinations of modal logics in general.

3 Tableau for LML

In (Artosi et al., 1996a, 1998, Governatori, 1997) a tableau-like proof system, called KEM, has been presented, and it has been proven to be able to cope with a wide variety of (normal) modal logics. KEM is based on D'Agostino and Mondadori's (1994) classical proof system KE, a combination of tableau and natural deduction inference rules which allows for a restricted ("analytic") use of the cut rule. The key feature of KEM, besides its being based neither on resolution nor on standard sequent/tableau inference techniques, is that it generates models and checks them using a label scheme for bookkeeping fibred models. In (Artosi et al., 1996b, Governatori, 1995, 1996, 1997) it has been shown how this formalism can be extended to handle various system of multi-modal logic with interaction axioms. The mechanism KEM uses in manipulating labels is close to semantic fibring (dovetailing).

[7] The modal dovetailing rule is really a necessitation rule. It says that if C is a wff built up from 'atomic' units of the form $\Box_p A_s$ and ordinary atoms P_k and $\Box_q$ is a modality *different* from $\Box_p$, then a limited necessitation rule holds: $\vdash C$ implies $\vdash \Box_q^d C$ for any natural number d.

3.1 Label Formalism

In this section we introduce the label formalism we shall use in the course of the paper. KEM uses *Labelled Signed Formulas*, where a labelled signed formula is an expression of the form SA, i, where A is a wff of the logic, S (the truth sign) is in $\{T, F\}$, and i is a label. Notice that the labels we are referring to in this section are not the labels of I. In the case of modal logic we have a type of labels corresponding to various modalities, and each set of atomic labels is partitioned into the set of variables and the set of constants. Formally

$$\Phi^{p\in I} = \Phi_C^{p\in I} \cup \Phi_V^{p\in I}$$

where $\Phi_C^{p\in I} = \{w_1^p, w_2^p, \dots\}$ and $\Phi_V^{p\in I} = \{W_1^p, W_2^p, \dots\}$. The set of constant world symbols and variable world symbols are respectively

$$\Phi_C = \bigcup_{p\in I} \Phi_C^{p\in I} \qquad \Phi_V = \bigcup_{p\in I} \Phi_V^{p\in I}$$

The set of labels $\Im$ is then defined inductively as follows:

$$\begin{aligned} \Im \quad = \quad & \bigcup_{1\leq k} \Im_k \text{ where } \Im_k, k \in \mathbb{N} : \\ & \Im_1 = \Phi_C \cup \Phi_V; \\ & \Im_2 = \Im_1 \times \Phi_C; \\ & \Im_{n+1} = \Im_1 \times \Im_n, (n > 1). \end{aligned}$$

According to the above definition a label is either a (i) an element of the set Φ_C, or (ii) an element of the set Φ_V, or (iii) a path term (k', k) where (iiia) $k' \in \Phi_C \cup \Phi_V$ and (iiib) $k \in \Phi_C$ or $k = (i', i)$ where (i', i) is a label. From now on we shall use $i, j, k, \dots$ to denote arbitrary labels.

Definition 12 *For any label $i = (k', k)$ we shall call k' the* head *of i, k the* body *of i, and denote them by $h(i)$ and $b(i)$ respectively.*

Notice that these notions are recursive (they correspond to projection functions): if $b(i)$ denotes the body of i, then $b(b(i))$ will denote the body of $b(i)$, $b(b(b(i)))$ will denote the body of $b(b(i))$; and so on. We call each of $b(i)$, $b(b(i))$, etc., a *segment* of i. Let $s(i)$ denote any segment of i (obviously, by definition every segment $s(i)$ of a label i is a label); then $h(s(i))$ will denote the head of $s(i)$. We shall call a label i *restricted* if $h(i) \in \Phi_C$, otherwise *unrestricted.*

We shall conceive world labels as worlds (set of worlds) in a fibred (dovetailing) model, and path labels as worlds with the path leading to them starting from the actual world. For example a label such $i = (W_2^p, (w_2^q, w_0))$ denotes with respect to a fibred (dovetailing) model the set of worlds in a model for $\mathbf{L}_p$ accessible from the actual world $a^p = \mathbf{F}_p((w_2^q, w_0))$; moreover the world the fibring function has been applied to, i.e., (w_2^q, w_0), is a world accessible from a^q, which is the result of fibring the actual world w_0 with respect to a model of type $\mathcal{K}_q$, i.e., $a^q = \mathbf{F}_q(w_0)$. Since we are mainly interested in the dovetailing, and dovetailed models can be reduced to structures of the form $\langle W, R(p), h\rangle$, the label i turns out to represents the set of worlds accessible via $R(p)$ from a world accessible through $R(q)$ from the actual world w_0.

Definition 13 *For any label i, we define the length of i, $\ell(i)$, as the number of world-symbols in i, i.e., $\ell(i) = n \Leftrightarrow i \in \Im_n$. $s^n(i)$ will denote the segment of i of length n, i.e.,*

$s^n(i) = s(i)$ such that $\ell(s(i)) = n$. We shall use $h^n(i)$ as an abbreviations for $h(s^n(i))$. Notice that $h(i) = h^{\ell(i)}(i)$.

DEFINITION 14 *For any label $i, l(i) > n$, we define the* countersegment-n *of i, as follows:*

$$c^n(i) = h(i) \times (\cdots \times (h^k(i) \times (\cdots \times (h^{n+1}(i), w_0)))) \quad (n < k < l(i))$$

where w_0 is a dummy label, i.e., a label not appearing in i (the context in which such a notion occurs will tell us what w_0 stands for; in most cases it will denote an actual world).

The countersegment-n defines what remains of a given label after having identified the segment of length n with a "dummy" label w_0. The appropriate dummy label will be specified in the applications where such a notion is used. However, it can be viewed also as an independent atomic label. In the context of fibring w_0 can be thought as denoting the actual world obtained via the fibring function from the world denoted by $s^n(i)$.

EXAMPLE 15 Given the label $i = (w_4, (W_3, (w_3, (W_2, w_1))))$, according to the above definitions its length $\ell(i)$ is 5, its segment of length 3 is $s^3(i) = (w_3, (W_2, w_1)$, and the relative countersegment-3 is $c^3(i) = (w_4, (W_3, w_0))$, where $w_0 = s^3(i) = (w_3, (W_2, w_1)$.

To clarify the notion of countersegment, which will be used frequently in the course of the present work, we present, in the following table the list of the segments of i in the left-hand column and the relative countersegments in the right-hand column.

$s^1(i) = w_1$	$c^1(i) = (w_4, (W_3, (w_3, (W_2, w_0))))$
$s^2(i) = (W_2, w_1)$	$c^2(i) = (w_4, (W_3, (w_3, w_0)))$
$s^3(i) = (w_3, (W_2, w_1))$	$c^3(i) = (w_4, (W_3, w_0))$
$s^4(i) = (W_3, (w_3, (W_2, w_1)))$	$c^4(i) = (w_4, w_0)$
$s^5(i) = i$	$c^5(i) = w_0$

So far we have provided definitions about the structure of the labels without regard of the elements they are made of. The following definition will be concerned with the type of world symbols occurring in a label.

DEFINITION 16 *We say that a label i is* p-preferred *iff $h(i) \in \Phi^p$.*

DEFINITION 17 *We say that a label i is* p-pure *iff each segment of i of length $n > 1$ is p-preferred. We shall use $\Im^p$ to denote the set of p-pure labels.*

3.2 Unifications

In the course of proofs labels are manipulated in a way closely related to the semantic of the logics under analysis. Labels are confronted and matched using a specialised logic dependent unification mechanism. The notion that two labels i and k unify means that the intersection of their denotations is not empty and that we can move to such a set of worlds, i.e., to the result of their unification.

According to the semantics each modality is evaluated in an appropriate model corresponding to a model in the class of models characterising the logic the modality corresponds to. Similarly we provide an unification for each logic, the unification characterising such a logic in KEM formalism, then we graft them into a single unification for the whole LML (HML).

3.2.1 Basic Unifications (Axiom Unifications)

We add a set of auxiliary unindexed atomic labels $\Phi^A = \{w_0, w_0', \dots\}$, that will be used in unifications and proofs. Intuitively they stand for distinguished worlds (actual worlds) in the various models. We define two substitutions, resp. dovetailing and fibring substitution, in the usual way as a mapping

$$\begin{aligned} \sigma^{\mathbf{D}} &: \Phi_V^p \longrightarrow \Im^p \cup \Phi^A \\ \sigma^{\mathbf{F}} &: \Phi_V^p \longrightarrow \Im^p \end{aligned}$$

where $\Im^p$ is the set of p-pure labels. Henceforth we use σ to mean indifferently, unless specified, either $\sigma^{\mathbf{D}}$ or $\sigma^{\mathbf{F}}$. For two labels i and k, and a substitution σ, if σ is a unifier of i and k then we shall say that i, k are *σ-unifiable*. We shall (somewhat unconventionally) use $(i,k)\sigma$ to denote both that i and k are σ-unifiable and the result of their unification. On this basis we define several specialised, logic-dependent notions of σ-unification corresponding to the axioms of the logic under analysis.

(σ^K) $\quad (i,k)\sigma^K = (i,k)\sigma$ if at least one of i and k is restricted, and $\forall n \leq \ell(i), (s^n(i), s^n(k))\sigma^K$

(σ^D) $\quad (i,k)\sigma^D = (i,k)\sigma$

$$(\sigma^T) \qquad (i,k)\sigma^T = \begin{cases} (s^{\ell(k)}(i), k)\sigma & \ell(i) > \ell(k), \text{ and} \\ & \forall n \geq \ell(k), (h^n(i), h(k))\sigma = (h(i), h(k))\sigma \\ (i, s^{\ell(i)}(k))\sigma & \ell(k) > \ell(i), \text{ and} \\ & \forall n \geq \ell(i), (h(i), h^n(k))\sigma = (h(i), h(k))\sigma \end{cases}$$

$$(\sigma^4) \qquad (i,k)\sigma^4 = \begin{cases} c^{\ell(i)}(k) & \ell(k) > \ell(i), h(i) \in \Phi_V \text{ and} \\ & w_0 = (i, s^{\ell(i)}(k))\sigma \\ c^{\ell(k)}(i) & \ell(i) > \ell(k), h(k) \in \Phi_V \text{ and} \\ & w_0 = (s^{\ell(k)}(i), k)\sigma \end{cases}$$

$$(\sigma^B) \qquad (i,k)\sigma^B = \begin{cases} (s^{\ell(i)-2n}(i), k)\sigma & \text{if } h(i) \in \Phi_V \text{ and} \\ & (h(i), h(k))\sigma = (h^{\ell(i)-2n}(i), h(k))\sigma \\ (i, s^{\ell(k)-2n}(k))\sigma & \text{if } h(k) \in \Phi_V \text{ and} \\ & (h(i), h(k))\sigma = (h(i), h^{\ell(k)-2n}(k))\sigma \end{cases}$$

Where $1 \leq n \leq V$, and $V = \ell(i) - m$, with m such that $\forall p, m \leq p \leq \ell(i), h^p(i) \in \Phi_V$.

$$(\sigma^5) \qquad (i,k)\sigma^5 = \begin{cases} ((h(i), h(k))\sigma; c^1(s^2(i))) & \ell(i) > 2, \ell(k) > 1, h(i) \in \Phi_V, \text{ or} \\ & h(i) = h(k) \in \Phi_C \\ (i,k)\sigma & \ell(i) = \ell(k) = 2 \\ ((h(i), h(k))\sigma; c^1(s^2(i))) & \ell(k) > 2, \ell(i) > 1, h(k) \in \Phi_V, \text{ or} \\ & h(i) = h(k) \in \Phi_C \end{cases}$$

where $w_0 = (s^1(i), s^1(k))\sigma$.

(σ^O) $\quad (i,k)\sigma^O = (c^2(i), c^2(k))\sigma^D$

where $w_0 = (s^2(i), s^2(i))\sigma^K$, and $O = \Box(\Box A \to \Diamond A)$.

EXAMPLE 18 For the notion of σ^T-unification, we consider the labels $i = (w_3, (W_1, w_1))$ and $k = (w_3, (W_2, (w_2, w_1)))$. Here $(W_2, w_3)\sigma = (w_3, w_3)\sigma$. Then i and k σ^T-unify to $(w_3, (w_2, w_1))$. This intuitively means that the world w_3, accessible from a sub-path $s(k) = (W_2, (w_2, w_1))$, after the deletion of W_2 from k, is accessible from any path i which turns out to denote the same world(s) as $s(k)$; in fact the step from w_2 to W_2 is irrelevant because of the reflexivity relation of the model. For the notion of σ^4-unification, take for example the labels $i = (W_3, (w_2, w_1))$ and $k = (w_5, (w_4, (w_3, (W_2, w_1))))$. Here $s^{\ell(i)}(k) = (w_3, (W_2, w_1))$. Then i and k σ^4-unify to $(w_5, (w_4, (w_3, (w_2, w_1))))$ since $(i, s^{\ell(i)}(k))\sigma = ((W_3, (w_2, w_1)), (w_3, (W_2, w_1)))\sigma$. This intuitively means that all the worlds accessible from a sub-path $s^{\ell(i)}(k)$ of k are accessible from any path i which leads to the same world(s) denoted by $s^{\ell(i)}(k)$.

3.2.2 High Unifications (Combined Unifications)

We are now able to combine the above unifications corresponding to the axioms characterising a logic into a single "high" unification which will be used for defining the unifications characterising the logic we are concerned with.

$$(\sigma^{A_1^p \cdots A_n^p}) \quad (i,k)\sigma^{A_1^p \cdots A_n^p} = \begin{cases} (i,k)\sigma^{A_1^p} & C_1^p \\ \vdots & \vdots \\ (i,k)\sigma^{A_n^p} & C_n^p \end{cases} \qquad p \in I$$

where $A_1^p \cdots A_n^p$ stand for the axioms characterising L_p, and $C_i, 1 \leq i \leq n$ are conditions varying from logic to logic. For example the high unification for the logic T is

$$(\sigma^{DT}) \quad (i,k)\sigma^{DT} = \begin{cases} (i,k)\sigma^{T} & \ell(i) \neq \ell(k) \\ (i,k)\sigma^{D} & \text{otherwise} \end{cases}$$

i.e., the combination of the unifications characterising the axioms of such a logic.

For OM, which is K plus $M = \Box(\Box A \to A)$, the deontic version of T, the corresponding high unification is

$$(\sigma^{OM}) \quad (i,k)\sigma^{OM} = \begin{cases} (i,k)\sigma^{T} & (s^2(i), s^2(k))\sigma^O \\ (i,k)\sigma^{O} & \text{otherwise} \end{cases}$$

3.2.3 Low Unification (Logic Unifications)

Combining recursively each high unification we obtain the σ_{L_p} unification for L_p as follows

$$(\sigma_{L_p}) \quad (i,k)\sigma_{L_p} = \begin{cases} (c^n(i), c^m(k))\sigma^{A_1^p \cdots A_n^p} \\ (i,k)\sigma^{A_1^p \cdots A_n^p} \end{cases}$$

where $w_0 = (s^n(i), s^m(k))\sigma_{L_p}$.

Although the above definition provides a general method for obtaining the unification characterising the appropriate logic, for particular systems it may be preferred to define a different unification reflecting peculiar properties. For example, for $S5$, which is characterised by the class of frames where the accessibility relation is an equivalence relation, it is convenient to define σ_{S5} as follows:

$$(\sigma_{S5}) \quad (i,k)\sigma_{S5} = \begin{cases} (h(i), h(k))\sigma & \min\{\ell(i), \ell(k)\} = 1 \\ ((h(i), h(k))\sigma, (s^1(i), s^1(k))\sigma) & \text{otherwise} \end{cases}$$

Due to the equivalence relation of the model, the path leading to a world is irrelevant. The only important condition is that the worlds denoted by the labels are in the same class of equivalence. This is achieved by the conditions on $s^1(i)$ and $s^1(k)$: they should be the same actual world.

The format of the unifications for the logics containing axioms obtained by prefixing $n\ \Box$ to one of the five axiom listed in section 1 is slightly more complex. We exemplify it by providing the unification for OM.

$$(\sigma_{OM}) \quad (i,k)\sigma_{OM} = \begin{cases} (c^n(i), c^m(k))\sigma^{OM} \\ (c^n(i), c^m(k))\sigma^{T} & n,m > 2 \\ (i,k)\sigma^{OM} \end{cases}$$

where $w_0 = (s^n(i), s^m(k))\sigma_{OM}$.

3.2.4 Fibred Unifications

As a Labelled Modal Logic is obtained by combining the logic L_p, $p \in I$ the corresponding unification is the combination of the σ_{L_p}-unifications characterising L_p.

$$(\sigma_{LML}) \quad (i,k)\sigma_{LML} = \begin{cases} \bigcup_{p\in I}(c^n(i), c^m(k))\sigma_{L_p} \\ \bigcup_{p\in I}(i,k)\sigma_{L_p} \end{cases}$$

where $w_0 = (s^n(i), s^m(k))\sigma_{LML}$, if $c^n(i), c^n(k)$ are p-pure, $p \in I$.

Similarly for HML

$$(\sigma^{p\in I}_{HML}) \quad (i,k)\sigma^{p\in I}_{HML} = \begin{cases} \bigcup_{p\in I}(c^n(i), c^m(k))\sigma_{L_p} \\ \bigcup_{p\in I}(i,k)\sigma_{L_p} \end{cases}$$

where $w_0 = (s^n(i), s^m(k))\sigma^{q\in I}_{HML}$ such that $p \prec q$, and $c^n(i), c^n(k)$ are p-pure.

Remark 19 It is worth noting that we have no constraints on the component logics; they may be combined logics themselves.

Example 20 In this example we provide the σ_{LML}-unification for ED. Let $e_1, e_2, \ldots$, $E_1, E_2, \ldots$, $d_1, d_2, \ldots$ and $D_1, D_2, \ldots$ denote, respectively, elements of Φ^1_C, Φ^1_V, Φ^2_C, and Φ^2_V. The unification for ED is defined as follows

$$(i,k)\sigma_{ED} = \begin{cases} (c^n(i), c^m(k))\sigma_{D45^1} \\ (i,k)\sigma_{D45^1} \\ (c^n(i), c^m(k))\sigma_{D^2} \\ (i,k)\sigma_{D^2} \end{cases}$$

where $w_0 = (s^n(i), s^m(k))\sigma_{ED}$, if $c^n(i)$ and $c^n(k)$ are p-pure, $p \in \{1,2\}$.

According to the above definition, the labels $i = (E_2, (e_1, (D_1, w_1)))$ and $j = (e_2, (d_1, w_1))$ σ_{ED}-unify. In fact $c^2(i) = (E_2, (e_1, w_0)$ and $c^2(j) = (e_2, w_0)$, σ_{D45^1}-unify, moreover $w_0 = ((D_1, w_1), (d_1, w_1))\sigma_{D^2}$. In contrast, if i had been $i' = (E_2, (d_2, (e_1, (D_1, w_1))))$, it would not have unified with j: $c^2(i')$ is not 2-pure.

It is immediate to see that the fibred unifications adopt the same strategy of fibring and dovetailing of sections 2.2 and 2.2.1. Each time we meet an atomic label of a different type we try to unify the segment with the appropriate unification and we identify the result with the actual world (w_0). At this point we use the unification for the logic corresponding to the label we have encountered.

3.3 Inference Rules

In displaying the rules of KEM we use Smullyan-Fitting (Fitting, 1983) $\alpha, \beta, \nu_p, \pi_p$, $p \in I$ unifying notation. Given a signed formula X, X^C denotes the *conjugate* of X, i.e., the result of changing the sign of X to its opposite; two LS-formulas X, i and X^C, k such that $(i,k)\sigma_L$ and will be called σ_L-*complementary*.

(α) $$\frac{\alpha, i}{\alpha_1, i} \qquad\qquad \frac{\alpha, i}{\alpha_2, i}$$

(β) $$\frac{\begin{array}{c}\beta, i\\ \beta_1^C, k\end{array}}{\beta_2, (i,k)\sigma_{L_I^\delta}}[(i,k)\sigma_{L_I^\delta}] \qquad\qquad \frac{\begin{array}{c}\beta, i\\ \beta_2^C, k\end{array}}{\beta_1, (i,k)\sigma_{L_I^\delta}}[(i,k)\sigma_{L_I^\delta}]$$

(ν_p) $$\frac{\nu_p, i}{\nu_0, (m,i)}[m \in \Phi_V^p \text{ and new}]$$

(π_p) $$\frac{\pi_p, i}{\pi_0, (m,i)}[m \in \Phi_C^p \text{ and new}]$$

(PB) $$\frac{}{X, i \qquad X^C, i}[i \text{ restricted}]$$

(PNC) $$\frac{\begin{array}{c}X, i\\ X^{C,k}\end{array}}{\times(i,k)\sigma_{L_I^\delta}}[(i,k)\sigma_{L_I^\delta}]$$

Here the α-rules are just the familiar linear branch-expansion rules of the tableau method, while the β-rules correspond to such common natural inference patterns as *modus ponens*, *modus tollens*, etc. (i, k, m stand for arbitrary labels). The rules for the modal operators are as usual. "m new" in the proviso for the ν_p- and π_p-rule means: m must not have occurred in any label yet used. Notice that in all inferences via an α-rule the label of the premise carries over unchanged to the conclusion, and in all inferences via a β-rule the labels of the premises must be σ_L-unifiable, so that the conclusion inherits their unification. PB (the "Principle of Bivalence") represents the (LS-version of the) semantic counterpart of the cut rule of the sequent calculus (intuitive meaning: a formula A is either true or false in any *given* world, whence the requirement that i should be restricted). PNC (the "Principle of Non-Contradiction") corresponds to the familiar branch-closure rule of the tableau method, saying that from the occurrence of a pair of σ_L-complementary formulas on a branch we may infer the closure ("$\times$") of the branch. The $(i,k)\sigma_L$ in the "conclusion" of PNC means that the contradiction holds "in the same world". Other logics might require additional rules in order to capture the full power of their semantics. See for example (Artosi et al., 1996b, Governatori, 1996). As usual with refutation methods a KEM-proof of A consists of a successful attempt to construct a counter model for A by assuming that A is false in some arbitrary model, which means that we assume that A is false in the actual world of the model.

EXAMPLE 21 In this example we show a KEM-proof of the formula we proved in section 1.1 using the dovetailed unifications for $T, D, K4$

1	$F[T][D][K4]A \rightarrow \langle D\rangle[K4][K4]A$	w_0
2	$T[T][D][K4]A$	w_0
3	$F\langle D\rangle[K4][K4]A$	w_0
4	$T[D][K4]A$	(W_1^T, w_0)
5	$F[K4][K4]A$	(W_2^D, w_0)
6	$T[K4]A$	$(W_3^D, (W_1^T, w_0))$
7	$F[K4]A$	$(w_2^{K4}, (W_2^D, w_0))$
8	TA	$(W_4^{K4}, (W_3^D, (W_1^T, w_0)))$
9	FA	$(w_3^{K4}(w_2^{K4}, (W_2^D, w_0)))$
10	$\times$	

Notice that 6 and 7 are not σ_{HML}-complementary: their labels do not unify; there is no way in which w_2^{K4} can be unified with another label. On the other hand 8 and 9 are σ_{HML}-complementary, in fact their labels can be analysed as follows:

$$(W_4^{K4}, w_0'') \qquad\qquad (w_3^{K4}(w_2^{K4}, w_0''))$$

which σ_{K4}-unify where

$$w_0'' = ((W_3^D, w_0'), (W_2^D, w_0'))\sigma_D \qquad\qquad w_0' = ((W_1^T, w_0), w_0)\sigma_T\ .$$

Soundness and completeness of KEM with respect to LML and HML can be proved by easy modifications of the proofs given in (Artosi et al., 1996a, Artosi et al., 1998, Governatori, 1997). In particular the following theorem has been proved.

THEOREM 22 *Let L_p be a normal modal logic obtained from the combination of the axioms D, T, 4, B, 5 and their recursive necessitations. Let $\mathcal{K}_p$ be the class of models characterising L_p.*

$$\models_{\mathcal{K}_p} A \iff \vdash_{KEM(L_p)} A\ .$$

In the course of the proof labels have been used to build appropriate models. Since the structure of the labels and unifications follows closely that of dovetailed and fibred models, we can repeat the same construction "grafting" (see (Gabbay, 1996a)) the models for each L_p through $\mathbf{F}$ into fibred and dovetailed models obtaining models for LML and HML. We can use these models for proving the following (see (Gabbay & Governatori, 1998) for a detailed proof):

THEOREM 23 *Let $L_p, p \in I$ be modal logics and let L_I the resulting combined logic. If $\models_{\mathcal{K}_p} A \iff \vdash_{KEM(L_p)} A$ then*

1. $\models_{L_I^F} A \iff \vdash_{KEM(L_I)} A$ *using* $\sigma^{\mathbf{F}}$;
2. $\models_{L_I^D} A \iff \vdash_{KEM(L_I)} A$ *using* $\sigma^{\mathbf{D}}$.

LML^F and HML^F denote labelled modal logics and hierarchical modal logics obtained by replacing necessitation and hierarchical necessitation by the modal fibring rule, and satisfying the conditions of theorem 8. From theorem 23 we obtain:

COROLLARY 24 *Let $\mathbf{L}$ be either an LML or an LML^F or a HML or a HML^F.*

$$\models_{\mathbf{L}} A \iff \vdash_{KEM(\mathbf{L})} A\ .$$

4 Final Remarks

In this work we provided a realization of Scott's idea by presenting a method for combining several logical systems into new ones, we have then shown how to adapt KEM to the resulting logics. Moreover we argued that LMLs and HMLs are a powerful tool for analysing normative reasoning. They allow us to split a phenomena, even complex ones like normative systems, into several components, each of them to be investigated with a specialised and suitable logic. We have only examined a few normal modal logics built upon classical propositional logic. But 1) the methodology we have used can be extended to other logics, such as the relevant deontic logics proposed by Goble (1998), using the techniques developed in (D'Agostino & Gabbay, 1994, D'Agostino & Gabbay, 1996, D'Agostino et al., 1996); 2) we believe KEM to be flexible enough to be adapted to combine a wide range of logics having relational semantics. Let us consider the following case. Makinson (1998) argues that a "realistic" system for deontic logic should take care of temporal parameters. On the other hand he develops an iterative construction to define conditional deontic notions. Such a construction gives raises to a particular non-monotonic deontic consequence relation which fails to provide an interpretation to nested norms (i.e., norms occuring in other norms). Gabbay (1995) shows that a conditional logic is nothing else than a non-monotonic consequence relation fibred with itself. But, it is well know that a non-monotonic consequence relation corresponds to the flat fragment of a conditional logic. Artosi and Governatori (1998) propose a tableau-like methodology for the flat fragment of conditional logics using KEM. So we can use such a methodology to reconstruct Makinson's account from a tableaux perspective, at this point we can fibre it with itself. Notice that the resulting system is able to deal with nested norms. At any point in the combination process we can insert the most appropriate temporal device. In this way it is possible to describe calculi for "realistic" deontic systems where temporal parameters are implemented. In the same way other relevant information, such as agency, priorities,. . . , can be incorporated in the system. Our methodology can be used to formulate an automated proof theory for the HML proposed by Cholvy & Cuppens (1998) to reason about conflicting codes.

We have not investigated applications (which will the subject of future works), even if the work on deontic defeasibility in (Artosi et al., 1996b) is a clear example of the potentiality of the framework.

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