Fibred Modal Tableaux
(preliminary report)

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Abstract. We describe a general and uniform tableau methodology for multi-modal logics arising from Gabbay's methodology of fibring and Governatori's tableau system KEM.

1 Introduction

In [3, 2, 17] a labelled tableau-like proof system for modal logic has been proposed. In [15, 16] it has been showed how such a system, called KEM, can be extended to deal with multi-modal logics. On the other hand Gabbay [11, 14] presented a general methodology for combining logical systems called fibring. Beckert and Gabbay [6] have analysed under which conditions a tableau system for a component logic can be plugged in a combined system. On the contrary, in the present work, we examine a case study: we develop a general and uniform tableau-like proof method for fibring (dovetailing) of modal logics. In particular we show how the label formalism used in KEM matches Gabbay's fibring for (multi-) modal logics. The resulting system enjoys several interesting properties: it is modular, in the sense that the proof system for each logic is developed on its own and it is reused in the combination; it is uniform, in the sense that each system has its own peculiarities, but the framework remains constant among the combined systems; it seems to be flexible enough to deal with other logics, as well as their combinations; finally, in spite of the logical jargon used and abused in describing the system, we claim its naturalness, in so far as the idea behind it is very simple and easy to grasp. Let L_1 and L_2 be two modal logics for which a tableau system exists. We start a tableau for a formula A of L_1, when we have to process a formula of L_2 we begin a new proof for it in the appropriate system. However we do not really need to start a separate tableau, but we graft it into the original one. The labels, due to their structure, will store nicely all the information about such an operation.

In this essay we shall stick ourselves with KEM based proof method, nevertheless the label formalism, and the combining methodology, can be used with almost whatever proof system.

The paper is organized as follows: in section 2 we shall introduce two basic methods for combining logical systems, namely fibring (2.1) and dovetailing (2.2); in section 3 we recall the basic tableau system KEM for modal logics;

we shall relate the techniques from the previous sections in order to provide a general and uniform tableau methodology for multi-modal logics.

2 Combining Modal Logics

The methodology of fibring allows us to combine arbitrary logical systems to form a new system in a uniform way "fibring" their models (for detailed expositions see [11–14]). The main idea of fibring is very simple, and provides a new concept of possible world semantics. Let us suppose we want to combine two modalities $\Box_1$ and $\Box_2$ characterised, respectively, by the classes of models $\mathcal{K}_1$ and $\mathcal{K}_2$. We know how to evaluate $\Box_1 A$ in $\mathcal{K}_1$, $\Box_2 A$ in $\mathcal{K}_2$ and propositional formulas in both. All we need is a method for evaluating $\Box_1$ (resp. $\Box_2$) w.r.t. $\mathcal{K}_2$ (resp. $\mathcal{K}_1$). Each time we have to evaluate a formula $\mathcal{A}$ of the form $\Box_2 A$ in a world in a model of $\mathcal{K}_1$ we associate, via the fibring function $\mathbf{F}$, to the world a model in $\mathcal{K}_2$, or its actual world, where we calculate the truth value of the formula. Formally

$$w \vDash_{\mathbf{m}\in\mathcal{K}_1} \Box_2 A \iff \mathbf{F}_{\mathbf{m}}(w) \vDash_{\mathbf{m}'\in\mathcal{K}_2} \Box_2 A$$

$\mathcal{A}$ holds in w iff it holds the model associated to W through the fibring function $\mathbf{F}$. But now we are in an appropriate model for evaluating $\mathcal{A}$.

In the next two sections we define two ways, fibring and dovetailing, in which semantics can be combined. We first explain the concepts by taking a simple example. Suppose we want to combine two modal logics L_1 and L_2. Let $\mathcal{K}_1, \mathcal{K}_2$ be the respective Kripke semantics of the logic. Let $\mathbf{m}$ be a model in $\mathcal{K}_1$ and let t be a possible world of $\mathbf{m}$. The semantic construction which combines the logics associates to t a model $\mathbf{n}$ in $\mathcal{K}_2$. The different methodologies of combination differ on the kind of model $\mathbf{n}$ we use. For *fibring* logics, we require that $\mathbf{n}$ be any model in $\mathcal{K}_2$. For *dovetailing*, we require that $\mathbf{n}$ be a model of $\mathcal{K}_2$ such that for any *atomic* P

$$t \vDash P \text{ iff } \mathbf{n} \vDash P$$

(i.e., the fibred model must agree with the values t gives to atoms).

First of all we have to introduce the language of the combined logic. Let $\mathbb{L}_p$ be the language of the logic L_p. The expressions E^p of the language $\mathbb{L}_p$ are built up using $\mathbb{L}_p$-constructors (connectives and operators) from a set of atomic units P^p. We schematically write $E^p(P^p_1, \ldots, P^p_n)$ to indicate that E^p is built up from the atoms $P^p_1, \ldots, P^p_n \in P^p$.

Definition 1. *The fibred language $\mathbb{L}_{(x_1,\ldots,x_n)}$ is defined as follows:*

- *Let $\mathbb{L}_{(p)}$ be $\mathbb{L}_p$, $p \in I$;*
- *Let $\bar{y}$ be $(q, y_1, \ldots, y_k)$, $q, y_1, \ldots, y_k \in I$ and $p \neq q$. Let $\mathbb{L}_{(p)*\bar{y}}$ be the family of all expressions of the form $\alpha \in \mathbb{L}_{(y_1,\ldots,y_k)}$ or $\alpha = E^p(P_1/A_1, \ldots, P_n/A_n)$ where $E^p(P_1, \ldots, P_n) \in \mathbb{L}_p$ and $A_1, \ldots, A_n$ are in $\mathbb{L}_{\bar{y}}$, and P_j/A_j indicates the substitution of of A_j to p_j in E^p;*
- $\mathbb{L}_I = \bigcup_{\bar{y}} \mathbb{L}_{\bar{y}}$.

2.1 Fibring Modal Logics

Let $L_p, p \in I$ be modal logics in the respective language $\mathbb{L}_p$, with $\Box_p, p \in I$, respectively. Let $\mathcal{K}_p$ be a class of models $\{\mathbf{m}_1^p, \mathbf{m}_2^p, \ldots\}$ for which L_p is complete. Each model $\mathbf{m}_n^p$ has the form (S, R, a, h) where S is the set of possible worlds, $a \in S$ is the actual world and $R \subseteq S^2$ is the accessibility relation. h is the assignment function, a binary function, giving a value $h(t, P) \in \{T, F\}$ for any $t \in S$ and atomic P. The actual world a plays a role in the semantic evaluation in the model, in so far as satisfaction in the model is defined as satisfaction at a. We can assume that the models satisfy the following condition:

$$S = \{x \mid \exists n \;\; aR^n x\} \; .$$

This assumption does not affect satisfaction in models because points not accessible from a by any power R^n of R do not affect truth values at a. Moreover we assume that all sets of possible worlds in any $\mathcal{K}_p$ are all pairwise disjoint, and that there are infinitely many isomorphic (but disjoint) copies of each model in $\mathcal{K}_p$. We use the notation $\mathbf{m}$ for a model and present it as $\mathbf{m} = (S^{\mathbf{m}}, R^{\mathbf{m}}, a^{\mathbf{m}}, h^{\mathbf{m}})$ and write $\mathbf{m} \in \mathcal{K}_p$, when the model $\mathbf{m}$ is in the semantics $\mathcal{K}_p$. Thus our assumption boils down to $\mathbf{m} \neq \mathbf{n} \Rightarrow S^{\mathbf{m}} \cap S^{\mathbf{n}} = \varnothing$. In fact a model can be identified by its actual world, i.e., $\mathbf{m} = \mathbf{n}$ iff $a^{\mathbf{m}} = a^{\mathbf{n}}$.

Definition 2. *A fibred model is a structure*

$$(W, W_{p, p \in I}, W_a, R, w_0, h, \mathbf{F})$$

where $W = \bigcup_{\mathbf{m} \in \cup_p \mathcal{K}_p} S^{\mathbf{m}}$; $W_p = \{a^{\mathbf{m}} \mid \mathbf{m} \in \mathcal{K}_p\}$; $W_a = \bigcup_p W_p$; $R = \bigcup_{\mathbf{m} \in \cup_p \mathcal{K}_p} R^{\mathbf{m}}$; $w_0 \in W_a$ *is the actual world;* $h(t, q) = h^{\mathbf{m}}(t, q)$, *for the unique* $\mathbf{m}$ *such that* $t \in S^{\mathbf{m}}$; $\mathbf{F} : I \times W \mapsto W_p$, *is the fibring function.*

The fibring function $\mathbf{F}$ *is a function giving for each* p *and each* $w \in W$ *another point (actual world) in* W_p *as follows:*

$$\mathbf{F}_p(w) = \begin{cases} w & \text{if } w \in S^{\mathbf{m}} \text{ and } \mathbf{m} \in \mathcal{K}_p \\ \text{a value in } W_p, & \text{otherwise} \end{cases}$$

such that if $x \neq y$, *then* $\mathbf{F}_p(x) \neq \mathbf{F}_p(y)$. *Satisfaction is defined as follows with the usual truth tables for boolean connectives:*

$$t \vDash P \quad \text{iff } h(t, P) = T$$

$$t \vDash \Box_p A \text{ iff} \begin{cases} t \in \mathbf{m}^p \text{ and } \forall s (tRs \to s \vDash A) \\ t \in \mathbf{m}^q, \;\; p \neq q \text{ and } \mathbf{F}_p(t) \vDash \Box_p A \end{cases}$$

We say the model satisfies A *iff* $w_0 \vDash A$.

Theorem 1 (Completeness theorem for the fibred logic L_I^F)**.** *Let* $L_p, p \in I$ *be modal logics in the respective language* $\mathbb{L}_p$ *with classes of structures* $\mathcal{K}_p$ *and set of theorems* $\mathbf{T}_p$. *(i.e.,* $\mathbf{T}_p = \{A$ *of* $\mathbb{L}_p \mid A$ *is valid in all* $\mathcal{K}_p$ *models* $\}$*). We consider two options for fibring. The general option where the logics* L_p *are assumed to have pairwise disjoint sets of atomic proposition (referred to as disjoint fibring) and the ordinary fibring where all* L_p *share the same set of atoms. Let* $\mathbf{T}_I^F$ *be the following set of wffs of* L_I^F.

1. $\mathbf{T}_p \subseteq \mathbf{T}_I^F$, *for every* $p \in I$. *For the case of ordinary fibring of* n *logics, we need axioms 1a and 1b bellow. If we are fibring an infinite number of logics, 1a is not needed.*
   1a. *If* A *is a Boolean combination of atoms and* $\alpha_{\bar{y}_n}$ *is in the* $\bar{y}$*-th fibred language, then* $A \to \alpha\bar{y}_n \in \mathbf{T}_{\bar{y}}$ *implies* $A \to \bigvee_n \alpha\bar{y}_n \in \mathbf{T}_I^F$, *where in* $\alpha_{\bar{y}_n}$ *every atom is in the scope of a modality.*
   1b. *If* $A(x_j) \in \mathbf{T}_p$ *then* $A(x_j/\Box_q\alpha_j) \in \mathbf{T}_I^F$, *for any* $\Box_q\alpha_j \in \mathbb{L}_q, q \in I$.
2. **Modal Fibring Rule:**
   If $\Box_p$ *is the modality of* L_p *and* $\Box_q$ *of* L_q, p, q *arbitrary* $p \neq q$ *and*

$$C = \bigwedge_{k=1}^{n} \Box_p A_k \to \bigvee_{k=1}^{m} \Box_p B_k \in \mathbf{T}_I^F$$

   then for all d, $\Box_q^d C \in \mathbf{T}_I^F$.
3. $\mathbf{T}_I^F$ *is the smallest set closed under 1, 2, modus ponens and substitution.*

Then $\mathbf{T}_I^F$ *is the set of all wffs of* L_I^F *valid in all the fibred structures of* L_I^F.

Proof. See [11, 14]. Notice that [11] deals with the case that the logics $L_p, p \in I$ have no atoms in common.

2.2 Dovetailing Modal Logics

Dovetailing arises in many applications where the fibred model at world t has the world t itself as its actual world. The notions of dovetailing results from that of fibring when for all $p \in I$ and for all atomic P

$$h(t, P) = h(\mathbf{F}_p(t), P) \ .$$

In such a case we can identify the actual world of the model fibred at t, $\mathbf{F}_p(t)$, with t. The fibring function $\mathbf{F}$ is no longer needed, since we identified t with $\mathbf{F}_p(t)$.

Let $L_p, p \in I$ be modal logics with $\mathcal{K}_p$ the class of models for L_p. Let L_I^D (the dovetailing combination of $L_p, p \in I$) be defined semantically through the class of all (dovetailed) models of the form (W, R, a, h), where W is a set of worlds, $a \in W$, h is an assignment as before, and for each $p \in I, R(p) \subseteq W \times W$. We require that for each p $(W, R(p), a, h)$ is a model in $\mathcal{K}_p$. We further require the following:
Let $t \in W$ be such that there exist $n_1, \ldots, n_k$ and $p_1, \ldots, p_k$ such that $aR^{n_1}(p_1) \circ R^{n_2}(p_2) \ldots \circ R^{n_k}(p_k)t$ holds.

We define the notion of $w \vDash A$ by induction.

- $w \vDash P$ iff $h(w, P) = T$ for P atomic.
- $w \vDash \Box_p A$ if for all $y \in W$, such that $wR(p)y$ we have $y \vDash A$.
- $\vDash A$ iff for all models and actual worlds $a \vDash A$.

Theorem 2 (Completeness theorem for the dovetailed logic L_I^D). *Let $L_p, p \in I$ be modal logics with semantical classes of structures $\mathcal{K}_p$ and set of theorems $\mathbf{T}_p$. Let $\mathbf{T}_I^D$ be the following set of wffs of L_I^D.*

1. $\mathbf{T}_p \subseteq \mathbf{T}_I^D$, *for every $p \in I$.*
2. **Modal Dovetailing Rule:**
   If $\Box_p$ is the modality of L_p and $\Box_q$ that of L_q, where p, q are arbitrary with $p \neq q$, and

$$C = \bigwedge_{k=1}^{n} \Box_p A_k \wedge \bigwedge_{k=1}^{m} \Diamond_p \neg B_k \rightarrow \bigvee_{k=1}^{r} P_k \in \mathbf{T}_I^D \ ,$$

   then for all d $\Box_q^d C \in \mathbf{T}_I^D$. Where P_k are atoms or their negations, and $P_1, \ldots, P_r$ list all the atoms or their negations appearing in any A_k or $B_k, k = 1, 2, \ldots$.
3. $\mathbf{T}_I^D$ *is the smallest set closed under 1, 2, modus ponens and substitution.*

Then $\mathbf{T}_I^D$ is the set of all wffs of L_I^D valid in all the dovetailed structures of L_I^D.

Proof. See [11].

3 Labelled Tableaux for Modal Logics

In [1, 2, 17] a tableau-like proof system, called KEM, has been presented, and it has been proven to be able to cope with a wide variety of (normal) modal logics. KEM is based on D'Agostino and Mondadori's [9] classical proof system KE, a combination of tableau and natural deduction inference rules which allows for a restricted ("analytic") use of the cut rule. The key feature of KEM, besides its being based neither on resolution nor on standard sequent/tableau inference techniques, is that it generates models and checks them using a label scheme for bookkeeping fibred models. In [5, 15–17] it has been shown how this formalism can be extended to handle various systems of multi-modal logic with interaction axioms. The mechanism KEM uses in manipulating labels is close to semantic fibring (dovetailing).

3.1 Label Formalism

In this section we introduce the label formalism we shall use in the course of the paper. KEM uses *Labelled Signed Formulas*, where a labelled signed formula is an expression of the form SA, i, where A is a wff of the logic, S (the truth sign) is in $\{T, F\}$, and i is a label. Notice that the label we are referring in this section are not the labels of I. In the case of modal logic we have a type of labels corresponding to various modalities, and each set of atomic labels is partitioned into the set of variables and the set of constants. Formally

$$\Phi^{p \in I} = \Phi_C^{p \in I} \cup \Phi_V^{p \in I}$$

where $\Phi_C^{p \in I} = \{w_1^p, w_2^p, \dots\}$ and $\Phi_V^{p \in I} = \{W_1^p, W_2^p, \dots\}$. The set of constant world symbols and variable world symbols are respectively

$$\Phi_C = \bigcup_{p \in I} \Phi_C^{p \in I} \qquad \Phi_V = \bigcup_{p \in I} \Phi_V^{p \in I}$$

The set of labels $\Im$ is then defined inductively as follows:

$$\begin{aligned} \Im = \bigcup_{1 \le k} \Im_k \text{ where } \Im_k, k \in \mathbb{N} : \\ \Im_1 = \Phi_C \cup \Phi_V; \\ \Im_2 = \Im_1 \times \Phi_C; \\ \Im_{n+1} = \Im_1 \times \Im_n, (n > 1). \end{aligned}$$

According to the above definition a label is either a (i) an element of the set Φ_C, or (ii) an element of the set Φ_V, or (iii) a path term (k', k) where (iiia) $k' \in \Phi_C \cup \Phi_V$ and (iiib) $k \in \Phi_C$ or $k = (i', i)$ where (i', i) is a label. From now on we shall use $i, j, k, \dots$ to denote arbitrary labels.

Definition 3. *For any label $i = (k', k)$ we shall call k' the* head *of i, k the* body *of i, and denote them by $h(i)$ and $b(i)$ respectively.*

Notice that these notions are recursive (they correspond to projection functions): if $b(i)$ denotes the body of i, then $b(b(i))$ will denote the body of $b(i)$, $b(b(b(i)))$ will denote the body of $b(b(i))$; and so on. We call each of $b(i)$, $b(b(i))$, etc., a *segment* of i. Let $s(i)$ denote any segment of i (obviously, by definition every segment $s(i)$ of a label i is a label); then $h(s(i))$ will denote the head of $s(i)$. We shall call a label i *restricted* if $h(i) \in \Phi_C$, otherwise *unrestricted*. We shall conceive worlds labels as worlds (set of worlds) in a fibred (dovetailing) model, and path labels as worlds with the path leading to them starting from the actual world. For example a label such $i = (W_2^p, (w_2^q, w_0))$ denotes with respect to a fibred (dovetailing) model the set of worlds in a model for $\mathbf{L}_p$ accessible from the actual world $a^p = \mathbf{F}_p((w_2^q, w_0))$; moreover, the world the fibring function has been applied to (i.e., (w_2^q, w_0)) is a world accessible from a^q, which is the result of fibring the actual world (w_0) with respect to a model of type $\mathcal{K}_q$, i.e., $a^q = \mathbf{F}_q(w_0)$. On the other hand dovetailed models can be reduced to structures of the form $\langle W, R(p), h \rangle$, the label i turns out to represents the set of worlds accessible via $R(p)$ from a world accessible through $R(q)$ from the actual world w_0. Notice that the structure of the labels allows us to represent without ambiguity chains of accessible worlds and their fibrings, i.e., chains of fibred models.

Definition 4. *For any label i, we define the length of i, $\ell(i)$, as the number of world-symbols in i, i.e., $\ell(i) = n \Leftrightarrow i \in \Im_n$. $s^n(i)$ will denote the segment of i of length n, i.e., $s^n(i) = s(i)$ such that $\ell(s(i)) = n$. We shall use $h^n(i)$ as an abbreviations for $h(s^n(i))$. Notice that $h(i) = h^{\ell(i)}(i)$.*

Definition 5. *For any label $i, l(i) > n$, we define the* countersegment-n *of i, as follows:*

$$c^n(i) = h(i) \times (\cdots \times (h^k(i) \times (\cdots \times (h^{n+1}(i), w_0))))(n < k < l(i))$$

where w_0 is a dummy label, i.e., a label not appearing in i, the context in which such a notion occurs will tell us what w_0 stands for. In most cases it will denote the actual world.

The countersegment-n defines what remains of a given label after having identified the segment of length n with a "dummy" label w_0. The appropriate dummy label will be specified in the applications where such a notion is used. However, it can be viewed also as an independent atomic label. In the contest of fibring w_0 can be thought as denoting the actual world obtained via the fibring function from the world denoted by $s^n(i)$.

Example 1. Given the label $i = (w_4, (W_3, (w_3, (W_2, w_1))))$, according to the above definitions its length $\ell(i)$ is 5, its segment of length 3 is $s^3(i) = (w_3, (W_2, w_1))$, and the relative countersegment-3 is $c^3(i) = (w_4, (W_3, w_0))$, where $w_0 = s^3(i) = (w_3, (W_2, w_1))$.

So far we have provided definitions about the structure of the labels without regard of the elements they are made of. The following definitions will be concerned with the type of world symbols occurring in a label.

Definition 6. *We say that a label i is* p-preferred *iff $h(t) \in \Phi^p$.*

Definition 7. *We say that a label i is* p-pure *iff each segment of i of length $n > 1$ is p-preferred, and we shall use $\Im^p$ to denote the set of p-pure labels.*

3.2 Unifications

In the course of proofs labels are manipulated in a way closely related to the semantic of the logics under analysis. Labels are confronted and matched using a specialised logic dependent unification mechanism. The notion of two labels i and k unify means that the intersection of their denotations is not empty and that we can move to such a set of worlds, i.e., to the result of their unification.

According to the semantics each modality is evaluated in an appropriate model corresponding to a model in the class of models characterising the logic the modality corresponds to. Similarly we provide an unification for each logic, the unification characterising such a logic in KEM formalism, then we graft them into a single unification for the whole L_I^δ.

Basic Unifications (Axiom Unifications) We add a set of auxiliary unindexed atomic labels $\Phi^A = \{w_0, w'_0, \ldots\}$, that will be used in unifications and proofs. Intuitively they stand for distinguished worlds (i.e., actual worlds) in the various models. We define two substitutions, resp. dovetailing and fibring substitution, in the usual way as a mapping

$$\sigma^{\mathbf{D}} : \Phi_V^p \longrightarrow \Im^p \cup \Phi^A$$
$$\sigma^{\mathbf{F}} : \Phi_V^p \longrightarrow \Im^p$$

where $\Im^p$ is the set of p-pure labels. Henceforth we use σ to mean indifferently, unless specified, either $\sigma^{\mathbf{D}}$ or $\sigma^{\mathbf{F}}$. For two labels i and k, and a substitution σ, if σ is a unifier of i and k then we shall say that i, k are *σ-unifiable*. We shall (somewhat unconventionally) use $(i, k)\sigma$ to denote both that i and k are σ-unifiable and the result of their unification. As a case study we choose the normal modal logics arising from the combination of the axioms On this basis we may define several specialised, logic-dependent notions of σ-unification. As a case study we choose the normal modal logics arising from the combination of the axioms K, D, T, 4, B and 5. Notice that the unifications listed below mimic the conditions on the accessibility relation corresponding to the appropriate axiom (see bellow for an explanation)

$$(i,k)\sigma^K = (i,k)\sigma \text{ if at least one of } i \text{ and } k \text{ is restricted, and} \\ \forall n \leq \ell(i), (s^n(i), s^n(k))\sigma^K \tag{σ^K}$$

$$(i,k)\sigma^D = (i,k)\sigma \tag{σ^D}$$

$$(i,k)\sigma^T = \begin{cases} (s^{\ell(k)}(i), k)\sigma & \ell(i) > \ell(k), \text{ and} \\ & \forall n \geq \ell(k), (h^n(i), h(k))\sigma = (h(i), h(k))\sigma \\ (i, s^{\ell(i)}(k))\sigma & \ell(k) > \ell(i), \text{ and} \\ & \forall n \geq \ell(i), (h(i), h^n(k))\sigma = (h(i), h(k))\sigma \end{cases} \tag{σ^T}$$

$$(i,k)\sigma^4 = \begin{cases} c^{\ell(i)}(k) & \ell(k) > \ell(i), h(i) \in \Phi_V \text{ and} \\ & w_0 = (i, s^{\ell(i)}(k))\sigma \\ c^{\ell(k)}(i) & \ell(i) > \ell(k), h(k) \in \Phi_V \text{ and} \\ & w_0 = (s^{\ell(k)}(i), k)\sigma \end{cases} \tag{σ^4}$$

$$(i,k)\sigma^B = \begin{cases} (s^{\ell(i)-2n}(i), k)\sigma & \text{if } h(i) \in \Phi_V \text{ and} \\ & (h(i), h(k))\sigma = (h^{\ell(i)-2n}(i), h(k))\sigma \\ (i, s^{\ell(k)-2n}(k))\sigma & \text{if } h(k) \in \Phi_V \text{ and} \\ & (h(i), h(k))\sigma = (h(i), h^{\ell(k)-2n}(k))\sigma \end{cases} \tag{σ^B}$$

Where $1 \leq n \leq V$, and $V = \ell(i) - m$, with m such that $\forall x, m \leq x \leq \ell(i), h^x(i) \in \Phi_V$.

$$(i,k)\sigma^5 = \begin{cases} ((h(i),h(k))\sigma; c^1(s^2(i))) & \ell(i) > 2, \ell(k) > 1, h(i) \in \Phi_V, \text{ or} \\ & h(i) = h(k) \in \Phi_C \\ (i,k)\sigma & \ell(i) = \ell(k) = 2 \\ ((h(i),h(k))\sigma; c^1(s^2(i))) & \ell(k) > 2, \ell(i) > 1, h(k) \in \Phi_V, \text{ or} \\ & h(i) = h(k) \in \Phi_C \end{cases} \qquad (\sigma^5)$$

where $w_0 = (s^1(i), s^1(k))\sigma$.
For the notion of σ^T-unification, take for example the labels $i = (w_3, (W_1, w_1))$ and $k = (w_3, (W_2, (w_2, w_1)))$. Here $(W_2, w_3)\sigma = (w_3, w_3)\sigma$. Then i and k σ^T-unify to $(w_3, (w_2, w_1))$. This intuitively means that the world w_3, accessible from a sub-path $s(k) = (W_2, (w_2, w_1))$, after the deletion of W_2 from k, is accessible from any path i which turns out to denote the same world(s) as $s(k)$; in fact the step from w_2 to W_2 is irrelevant because of the reflexivity relation of the model. For the notion of σ^4-unification, take for example the labels $i = (W_3, (w_2, w_1))$ and $k = (w_5, (w_4, (w_3, (W_2, w_1))))$. Here $s^{\ell(i)}(k) = (w_3, (W_2, w_1))$. Then i and k σ^4-unify to $(w_5, (w_4, (w_3, (w_2, w_1))))$ since $(i, s^{\ell(i)}(k))\sigma = ((W_3, (w_2, w_1)), (w_3, (W_2, w_1)))\sigma$. This intuitively means that all the worlds accessible from a sub-path $s^{\ell(i)}(k)$ of k are accessible from any path i which leads to the same world(s) denoted by $s^{\ell(i)}(k)$.

High Unifications (Combined Unifications) We are now able to combine the above unifications corresponding to the axiom characterising a logic into a single "high" unification which will be used for defining the unifications characterising the logic we are concerned with.

$$(i,k)\sigma^{A_1^p \cdots A_n^p} = \begin{cases} (i,k)\sigma^{A_1^p} & C_1^p \\ \vdots & \vdots \\ (i,k)\sigma^{A_n^p} & C_n^p \end{cases} \quad p \in I \qquad (\sigma^{A_1^p \cdots A_n^p})$$

where $A_1^p \cdots A_n^p$ stand for the axioms characterising L_p and $C_i, 1 \leq i \leq n$ are conditions varying from logic to logic. For example the high unification for the logic T is

$$(i,k)\sigma^{DT} = \begin{cases} (i,k)\sigma^T & \ell(i) \neq \ell(k) \\ (i,k)\sigma^D & \text{otherwise} \end{cases} \qquad (\sigma^{DT})$$

i.e., the combination of the unifications corresponding to the axioms characterizing such a logic.

Low Unification (Logic Unifications) Combining recursively each high unification we obtain the σ_{L_p} unification for L_p as follows

$$(i,k)\sigma_{L_p} = \begin{cases} (c^n(i), c^m(k))\sigma^{A_1^p \cdots A_n^p} \\ (i,k)\sigma^{A_1^p \cdots A_n^p} \end{cases} \qquad (\sigma_{L_p})$$

where $w_0 = (s^n(i), s^m(k))\sigma_{L_p}$.

Although the above definition provides a general method for obtaining the unification characterising the appropriate logic, for particular systems it may be preferred to define a different unification reflecting peculiar properties; for example, for $S5$ which is characterised by the class of frames where the accessibility relation is an equivalence relation, it is convenient to define σ_{S5} as follows:

$$(i,k)\sigma_{S5} = \begin{cases} (h(i), h(k))\sigma & \min\{\ell(i), \ell(k)\} = 1 \\ ((h(i), h(k))\sigma, (s^1(i), s^1(k))\sigma) & \text{otherwise} \end{cases} \qquad (\sigma_{S5})$$

Due to the equivalence relation the path leading to a world is irrelevant. The only important conditions is that the world denoted by the labels are in the same class of equivalence; this is achieved by the conditions on $s^1(i)$ and $s^1(k)$.

Fibred Unification As a Labelled Modal Logic is obtained by combining the logic L_p, $p \in I$ the corresponding unification is the combination of the σ_{L_p}-unifications characterising L_p.

$$(i,k)\sigma_{L_I^\delta} = \begin{cases} \bigcup_{p \in I} (c^n(i), c^m(k))\sigma_{L_p} \\ \bigcup_{p \in I} (i,k)\sigma_{L_p} \end{cases} \qquad (\sigma_{L_I^\delta})$$

where $w_0 = (s^n(i), s^m(k))\sigma_{L_I^\delta}$, if $c^n(i), c^n(k)$ are p-pure, $p \in I$.

We state now, without proving it, a property of $\sigma_{L_I^\delta}$-unifications.

Theorem 3. *$\forall i, j \in \Im$, if $(i,j)\sigma_{L_I^\delta}$, then $(i, (i,j)\sigma_{L_I^\delta})\sigma_{L_I^\delta}$ and $((i,j)\sigma_{L_I^\delta}, j)\sigma_{L_I^\delta}$.*

Proof. Since $\sigma_{L_I^\delta}$ has been defined as the combination of σ_{L_p}-unifications, but each of them works on p-pure labels and there are no interactions among them, we just have to prove the property for each σ_{L_p}; but such a proof has been given in [2, 17].

Remark 1. It is worth noting that we have no constraints on the component logics; they may be combined logics themselves.

3.3 Inference Rules

In displaying the rules of KEM we shall use Smullyan-Fitting [10] $\alpha, \beta, \nu_p, \pi_p$, $p \in I$ unifying notation. Given a signed formula X, X^C denotes the *conjugate* of X, i.e., the result of changing the sign of X to its opposite; two LS-formulas X, i and X^C, k such that $(i,k)\sigma_{L_I^\delta}$ and will be called *$\sigma_{L_I^\delta}$-complementary*.

$$\frac{\alpha, i}{\alpha_1, i} \qquad\qquad \frac{\alpha, i}{\alpha, i} \qquad (\alpha)$$

$$\frac{\begin{array}{c}\beta, i\\ \beta_1^C, k\end{array}}{\beta_2, (i,k)\sigma_{L_I^\delta}}[(i,k)\sigma_{L_I^\delta}] \qquad \frac{\begin{array}{c}\beta, i\\ \beta_2^C, k\end{array}}{\beta_1, (i,k)\sigma_{L_I^\delta}}[(i,k)\sigma_{L_I^\delta}] \tag{β}$$

$$\frac{\nu_p, i}{\nu_0, (m,i)}[m \in \Phi_V^p \text{ and new}] \tag{ν_p}$$

$$\frac{\pi_p, i}{\pi_0, (m,i)}[m \in \Phi_C^p \text{ and new}] \tag{π_p}$$

$$\frac{}{X, i \qquad X^C, i}[i \text{ restricted}] \tag{PB}$$

$$\frac{\begin{array}{c}X, i\\ X^{C,k}\end{array}}{\times(i,k)\sigma_{L_I^\delta}}[(i,k)\sigma_{L_I^\delta}] \tag{PNC}$$

Here the α-rules are just the familiar linear branch-expansion rules of the tableau method, while the β-rules correspond to such common natural inference patterns as *modus ponens*, *modus tollens*, etc. (i, k, m stand for arbitrary labels). The rules for the modal operators are as usual. "m new" in the proviso for the ν_p- and π_p-rule means: m must not have occurred in any label yet used. Notice that in all inferences via an α-rule the label of the premise carries over unchanged to the conclusion, and in all inferences via a β-rule the labels of the premises must be $\sigma_{L_I^\delta}$-unifiable, so that the conclusion inherits their unification. PB (the "Principle of Bivalence") represents the (LS-version of the) semantic counterpart of the cut rule of the sequent calculus (intuitive meaning: a formula A is either true or false in any *given* world, whence the requirement that i should be restricted). PNC (the "Principle of Non-Contradiction") corresponds to the familiar branch-closure rule of the tableau method, saying that from the occurrence of a pair of $\sigma_{L_I^\delta}$-complementary formulas on a branch we may infer the closure ("$\times$") of the branch. The $(i,k)\sigma_{L_I^\delta}$ in the "conclusion" of PNC means that the contradiction holds "in the same world". Other logics might require additional rules in order to capture the full power of their semantics. See for example [5, 16]. As usual with refutation methods a KEM-proof of A consists of a successful attempt to construct a counter model for A by assuming that A is false in some arbitrary model, which means that we assume that A is false in the actual world of the model.

Theorem 4. *If a KEM tree closes it closes atomically.*

Proof. A closed KEM tree means that each branch is closed, i.e., it contains two $\sigma_{L_I^\delta}$-complementary formulas A, i and A^C, j.

$$\begin{array}{c}\vdots\\ A, i\\ \vdots\\ A^C, j\\ \times\end{array}$$

We prove the theorem by induction of the complexity of the complementary formulas. If they are literals then the branch closes atomically.

If they are not literals let us examine their form: if A is of type α then A^C is of type β; moreover $\alpha_1 = \beta_1^C$ and $\alpha_2 = \beta_2^C$. We apply an α-rule on A, i, obtaining α_1, i and α_2, i. Since the relations just mentioned we can apply a β-rule w.r.t. A^C, j and α_n $(n = 1, 2)$, from which we derive $\beta_{3-n}, (i, j)\sigma_{L_I^\delta}$. At this point the branch contains α_n, i and $\beta_n, (i, j)\sigma_{L_I^\delta}$, which are $\sigma_{L_I^\delta}$-complementary, in so far as $(i, (i, j)\sigma_{L_I^\delta})\sigma_{L_I^\delta}$, see Lemma 3. If A is of type β we repeat the above reasoning applying the α-rule on A^C instead of A.

If A is of type ν, then A^C is of type π and $\nu_0 = \pi_0^C$. We apply a ν-rule on A, i and a π-rule on A^C, j obtaining $\nu_0, (W_n, i)$ and $\pi_0, (w_m, j)$, where W_n and w_m are new in the branch. The resulting formulas are $\sigma_{L_I^\delta}$-complementary due to the relationship among ν and π formulas and the fact that the labels obviously $\sigma_{L_I^\delta}$-unify. If A is of type π, then A^C is of type ν and we can repeat the same argument.

In the course of tableau proofs labels are used to build appropriate models. Since the structure of the labels and unifications follows closely that of dovetailed and fibred models, we can repeat the same construction "grafting" (see [11, 14]) the models for each L_p through $\mathbf{F}$ into fibred and dovetailed models obtaining models for L_I^δ.

Theorem 5. *Let $L_p, p \in I$ be modal logics and let L_I^δ the resulting combined logic. If $\models_{\mathcal{K}_p} A \iff \vdash_{\mathrm{KEM}(L_p)} A$ then*

1. $\models_{L_I^F} A \iff \vdash_{\mathrm{KEM}(L_I)} A$ *using* $\sigma^{\mathbf{F}}$;
2. $\models_{L_I^D} A \iff \vdash_{\mathrm{KEM}(L_I)} A$ *using* $\sigma^{\mathbf{D}}$.

Proof. We prove the theorem by showing a) the set $\mathbf{T}_I^\delta$ is a subset of set of formulas provable in KEM and b) KEM rules are sound with respect to fibred and dovetailed models.

We start proving a). According to Theorems 1 and 2 a formula is a fibred (dovetailed) theorem if either it is a theorem of a component, or has been obtained by one of the axioms, or has been derived from an application of modal fibring (dovetailing) rule or modus ponens. We have then to prove that such rules and axioms are derived in KEM.

By hypothesis $\mathbf{T}_p$ coincides with the set of formulas provable in KEM for L_p. Modus ponens is a derived rule of KE, the propositional fragment of KEM, for the proof see [9].

For axiom 1a (Theorem 1), by hypothesis $A \to \alpha_{\bar{y}_n}$ is a theorem of the fibred language $\bar{y}$, therefore TA, w_0 and $F\alpha_{\bar{y}_n}, w_0$ lead to a closed KEM-tree. Let us start now a KEM-tree with $A \to \bigvee_n \alpha_{\bar{y}_n}, w_0$ we obtain

$$\begin{array}{c} TA, w_0 \\ F \bigvee_n \alpha_{\bar{y}_n}, w_0 \\ F\alpha_{\bar{y}_n}, w_0 \\ \vdots \\ F\alpha_{\bar{y}_n}, w_0 \end{array}$$

At this point we can repeat the proof for $A \to \alpha\bar{y}_n$, closing thus the tree.

For axiom 1b (Theorem 1), by hypothesis $A(x_j)$ has a closed KEM-tree, which means that each branch τ is closed; Theorem 4 implies that each branch is atomically closed, therefore each branch contains two σ_{L_p}-complementary labelled signed literals, let us say x_τ, i_τ and x^C_τ, j_τ. We can now replace $\Box_q\alpha_j$ to x_τ obtaining $S\Box_q\alpha_j, i_\tau$ and $S^C\Box_q\alpha_j, j_\tau$. But the last two formulas are $\sigma_{L^F_I}$-complementary, then also in this case the tree is closed.

For modal and dovetailing rule, let us assume that C is a formula satisfying the conditions of the rules, then C has a closed KEM-tree. We show now that also the tree for $\Box^n_q C$ is closed.

$$\begin{array}{c} F\Box^n_q C, w_0 \\ F\Box^{n-1}_q C, (w^q_1, w_0) \\ \vdots \\ FC, (w^q_n, \dots (w^q_1, w_0)) \end{array}$$

All we have to do is to identify w'_0 with $(w^q_n, \dots (w^q_1, w_0))$ and we can repeat the proof for C, w'_0, in so far as $((w^q_n, \dots (w^q_1, w_0)), (w^q_n, \dots (w^q_1, w_0)))\sigma_{L^\delta_I}$.

For b) in the course of KEM-proofs, we generate labels according to the structure of the formulas involved, but, as we have already said, they also generate (counter)-models. The labels are intended to denote possible worlds and relations among them, and all the relevant information are recorded in the labels. So, to extract such information, we have to map labelled signed formulas to elements of fibred and dovetailed models. This is achieved with the help of three functions, namely g, r, and f. The function g will map labels to sets of possible worlds: a singleton for constants, a set of worlds (possibly empty) for variables, and an actual world for auxiliary labels. The accessibility relation R is assumed to be closed under specific conditions; but, we want to reconstruct it, through r, from the labels: path labels are intended to represent not only worlds, but also the chain of possible worlds leading to them. Finally, f, given an LS-formula, returns the evaluation of the formula with respect to the world(s) denoted by its label.

Let $\mathbf{m}^p = (S^p, R^p, a^p, h^p)$ be a model in $\mathcal{K}_p$ where: $S = \Phi^p_C$; R^p is a binary relation on S^p; $a^p \in \Phi^A$ and h^p is an evaluation function.

Let g be a function from $p^{\Im}$ to $\wp(S^p)$ thus defined:

$$g(i) = \begin{cases} \{h(i)_i\} & \text{if } h(i) \in \Phi^p_C \\ \{w_i \in S^p \mid g(b(i))Rw_i\} & \text{if } h(i) \in \Phi^p_V \\ S^p & \text{if } i \in \Phi^p_V \\ a^p & \text{if } i \in \Phi^A \end{cases}$$

It may be possible that two labels have the same head, but they denote different worlds, this is way we have indexed $h(i)$ with the label itself. However we shall drop the subscript, when this is harmless.

Let r be a function from the set of p-pure labels to R^p thus defined:

$$r(i) = \begin{cases} \emptyset & \text{if } l(i) = 1 \\ g(i^1)R^p g(i^2), \ldots, g(i^{n-1})R^p g(h(i)) & \text{if } l(i) = n > 1 \end{cases}$$

Let f be a function from LS-formulas to v thus defined:

$$f(SA, i) =_{def} h(w_j^p, A) = S$$

for all $w_j^p \in g(i)$.

Until now we have examined p-pure labels. Let i be a not p-pure label. It can be decomposed into p-pure labels as follows: let $n \in \mathbb{N}$ such that $\forall m > n$ $s^m(i)$ is p-preferred. The label $c^n(i)$ is p-pure. A not p-pure label can be conceived as a recursive fibring of p-pure sub-labels.

Let $\mathbf{m}$ be a structure as in definition 2, where the fibred function $\mathbf{F}$ is defined as follows:

$$\mathbf{F}_p(w_j) = g(w_0)$$

for each $w_j \in g(s^n(i))$, where 1) $w_0 = h^1(c^n(i))$ and 2) $c^n(i)$ is p-pure. Moreover we require that if $w_j \neq w_k$ then $\mathbf{F}_p(w_j) \neq \mathbf{F}_p(w_k)$. In the case of dovetailing we impose $\mathbf{F}_p(w_j) = w_j$

It is easy to see that $\mathbf{m}$ is a fibred or a dovetailed model for L_I^δ.

Lemma 1. *For any $i, k \in \Im$ if $(i, k)\sigma_{L_I^\delta}$ then $g(i) \cap g(k) \neq \emptyset$.*

Proof. The proof is by induction on the number of applications of σ_{L_p} in $\sigma_{L_I^\delta}$. First we have to prove the property for σ_{L_p} and therefore for $\sigma^{A_1 \cdots A_n}$. For a detailed proof see [2, 17].

Lemma 2. *For any $i, k \in \Im$, if $f(SA, i)$, $(i, k)\sigma_{L_I^\delta}$ then $f(SA, (i, k)\sigma_{L_I^\delta})$.*

Proof. Let us suppose, by contradiction, that the lemma does not hold, so the proof trivially follows from Lemma 1, and the definition of f.

The α-rules and PB are obviously sound rules in $\mathbf{m}$ in so far as they work locally. For the β-rules and PNC: by the hypothesis $(i, k)\sigma_{L_I^\delta}$, then, by Lemmas 1 and 2, there exists a world in $g((i, k)\sigma_{L_I^\delta})$, let us say w_j, where both β, β_n $(n = 1, 2)$ and β_{3-n} hold. This implies that β_n and β_n^C hold at w_j, which is a contradiction. The same argument can be applied in the case of PNC.

For the ν-rules. Let us suppose $\nu = T\Box_p A$ the case of $\nu = \Diamond_p A$ is identical. Let us suppose it does not hold, then for all $w_i \in g(i)$, $h(w_i, \Box_p A) = T$ and for some $w_m \in g((m, i))$, $h(A, w_m) = F$, where (m, i) is p-preferred. IF i is p-preferred, then $h(w_i, \Box_p A) = T$ implies $\forall w_j : w_i R^p w_j, h(w_j, A) = T$, By the definitions of g and r we know $w_i R^p w_m$, obtaining thus a contradiction. If i is not p-preferred, then $h(w_i, \Box_p A) = T$ iff $h(\mathbf{F}_p(w_i), \Box_p A) = T$. The label (m, i) is not p-pure, but (m, w_0) where $\mathbf{F}(w_i) = g(w_0)$ is. At this point we can repeat the same argument as before with w_0 instead of w_i. The proof for the π-rule is similar. This ends the proof of Theorem 5.

4 Final remarks

In the last few years we have witnessed a luxuriant growth of multi-dimensional systems in every field of logic, although only few seeds have been sown in the garden of proof theory [6–8].

In this work we have been mainly concerned with combinations of normal modal logics. It is easy to see that dovetailing of normal modal logics correspond to fusion [18]. However fibring is more general, and we have no constraints on the components. They may be as well combined logics. It's worth noting that, as far propositional normal modal logics are concerned, the system presented here satisfies all the conditions required for combined tableau calculi reported in [6]; in particular the condition stating "the labels that are part of tableau formulae represent words in models and do not contain other information". However KEM-based tableau calculi can be provided for different systems having Kripke-style semantics, e.g., quantified modal logics, non normal modal logics, conditional logics; but such systems require more information in the labels (see [1, 2, 4]). The unification scheme we have proposed is not sensitive of such additional burden, and can be applied also for their combination, in so far as it is closely related to the fibring function.

We have not studied multi-modal logics with interaction axioms specifying how they relate each other. Some instances of them have been studied in [15–17], where the key idea consists of providing either new substitutions and unifications or constraints on the already existing ones, both reflecting semantic properties of such axioms. Let us consider the axiom $\Box_1 A \rightarrow \Box_2 A$; as is well known it correspond to the condition $R_2 \subseteq R_1$. This is achieved in KEM by defining a substitution as follows: $\sigma_1 : \Phi_V^1 \mapsto \Im^1 \cup \Im^2$; sometimes special inference rules may be required (see [16]). Similarly non-normal modal logics can be dealt with by weakening the substitutions, e.g., $\sigma : \Phi_V \mapsto \Im^*$ where $\Im^* \subseteq \Im$. At the light of what we have already said the study of the proof theory of multi-modal logics reduces to the study of the components.

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